Master thesis

Computer-assisted two-loops renormalization of six-dimensional scalar effective field theories

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Abstract

In this master's thesis, computer-algebra codes are written to automate the analytical calculations emerging from the renormalization procedure of scalar effective field theories, which are considered up to 6-dimensional operators and up to 2-loops Feynman diagrams. The computer programs are applied to determine the beta functions and the anomalous dimensions of ϕ^4-theory, of a single scalar field effective theory and of a two interacting scalar fields effective theory, for which a complete characterization of its Green's basis and of its physical basis is given.

In the first part of the thesis, known regularization and renormalization techniques are introduced. In particular, a formal regularization framework is derived and a computationally efficient renormalization approach, based on bare quantities, is presented. Then, effective field theories are introduced along with their construction method through an operator product expansion. Redundancies among the operators of the latter will lead to the definition of Green's and physical operator bases, which in turn will lead to the powerful effective beta functions theorem. In the second part of the thesis, a symbolic evaluation algorithm of the integrals depicted by Feynman diagrams is thoughtfully derived, so that their evaluation can be implemented in computer-algebra systems. The algorithm is based on the techniques of tensor reduction, power reduction to a set of master integrals and dimensional shifts, all of which are described and derived to be readily applied to 1 and 2 loops diagrams whose fields have equal masses. A list of amplitudes, verified analytically by hand and to be later used in the renormalization procedures, is then generated by the implemented program. Moreover, a symbolic loop order counting technique, independent of the graphical representation of the diagrams and therefore handy for renormalization, is presented and proved. In the last part of the thesis, the computer-assisted renormalization procedure is carried out on the three scalar field theories of interest. For ϕ^4-theory, all calculations are also carefully replicated by hand. For the two other effective field theories, their Green's and physical bases are derived, along with the integrations by parts and field redefinitions needed to remove all their redundant operators. Finally, their beta functions and anomalous dimensions are obtained, at 2 loops precision for ϕ^4-theory and for the first effective field theory, and at 1 loop precision for the second effective field theory.

Contents

Chapter 1

Introduction

1.1 Motivation

The formulation of a **theory of everything** has been the ultimate goal of theoretical physicists starting from the breakthroughs achieved at the beginning of the last century. The hope was to find a universal theory able to describe every phenomenon in the Universe happening at every energy scale. However, to this day, all the unification attempts to reconcile this way the quantum fundamental interactions of the **Standard Model** with the deformed spacetime gravitational interaction predicted by General Relativity have failed. Even worse, new unexpected **Beyond the Standard Model** (**BSM**) phenomena emerged in the meantime, the most notable one being the existence of dark matter. As it became clear that the audacious quest of such a unified theory turned out to be unsuccessful, theoretical physicists shifted their ambitions towards the formulation of more attainables **theories of something**, nowadays known as **effective theories** [1, p.3]. The goal of these theories is not to describe everything in the most general possible way, but rather to describe selected phenomena in a restricted energy range. In this sense they are an effective description of reality, whose degrees of freedom are not meant to be fundamental but rather emergent from arbitrary choices, and they must be regarded as low-energy limits of more general theories, be them known or yet to be discovered [1, p.3]. The fact that effective theories only describe parts of reality doesn't jeopardize research of new physics, quite the contrary: because they merely are low-energy limits, they may be arbitrarily modified to account for previously neglected phenomena. In **effective field theories** (**EFTs**), these modifications take place at the Lagrangian level, where new terms may be added to the so-called **Operator Product Expansion** (**OPE**). Under this programme, in current research the Standard Model is elevated to an effective theory called the **Standard Model Effective Field Theory** (**SMEFT**), and its OPE is constructed out of an operator basis called the **Warsaw basis**.

To compare the theoretical predictions of the EFTs with the empirically measured evidences, physical observables must be computed. In field theories, the main quantities of interest are S-matrix elements, rendered finite after renormalizing their respective theories [2, p.2]. Said renormalization procedure removes the infinities from all the diverging n-point functions, which contain all Feynman diagrams up to a chosen loop order. Therefore, the bulk of renormalization consists of finding these infinities by explicitly evaluating all the regularized integrals encoded in the Feynman diagrams. While in principle this task can be analytically conducted fully by hand, as the loop order and the length of the OPE increase, the number of generated Feynman diagrams grows uncontrollably. This huge amount of Feynman diagrams makes it practically impossible to renormalize even simple EFTs past 1 loop order by hand, hence the necessity to automate the renormalization procedure through

computer-algebra codes, which can process hundreds of diagrams in an eye blink and without any algebra mistakes. So far, the SMEFT has been renormalized at 1 loop order [3]: in this thesis we lay down the theoretical and computational foundations required to renormalize the SMEFT at 2 loops order.

1.2 General outline

We now sketch the content of the thesis and how it will be presented.

In the first Chapter 2, all the employed conventions and notations are defined to render the text unambiguous. Moreover, all the recurrent mathematical formulas are presented and some are proved. A refresher on generating functionals is also given.

In Chapter 3, the regularization of diverging integrals and the renormalization of field theories are tackled in a general manner. In the first part of the Chapter, Dimensional Regularization is formally justified and it is readily applied to archetypal integrals, namely the tadpole integral and three variations of the sunset integral. In the second part of the Chapter, all the relevant quantities required for the renormalization procedure by counterterms are derived, most importantly the Renormalization Group Equations. Furthermore, we show an astute link between renormalized and bare n-point functions, which will allow us to circumvent the recalculation of Feynman diagrams and hence to avoid using the inconvenient renormalized free propagator.

In Chapter 4, EFTs are formally introduced along with an OPE construction recipe. Because the latter is in principle an infinite tower of operators, a cutoff prescription is devised: this will imply a new notion of renormalizability for EFTs and it will provide a systematic way to truncate EFTs results, notably through the power counting formula. Integrations by parts and field redefinitions are introduced to remove redundant operators from the OPEs, leading to the concepts of Green's and physical operator bases. Lastly, the effective beta functions theorem is presented, and it will later allow us to base all our Feynman diagrams calculations solely on the Feynman rules associated with the operators of the physical bases, significantly reducing the computational power required by renormalization.

In Chapter 5, a symbolic evaluation algorithm of the integrals depicted by Feynman diagrams is presented. The method is based on three main steps: tensor reduction, power reduction, and dimensional shifts. The first step of the algorithm strips away all the tensor indices of the loop momenta, potentially present in the integrand, to cast them away inside constant tensors, which therefore can be brought out of the integral. After this step, the integral is said to be scalar. The second step reduces, via integration by parts identities, general scalar integrals to a linear combination in terms of a restricted basis of scalar integrals, called the master integrals. This means that only an analytical evaluation of the master integrals is needed to compute any arbitrary Feynman diagram integral. The third step reduces the dimension of the master integrals of the previous linear combination, which might have been incremented during tensor reduction, down to the initial one. Due to its mechanical and repetitive nature, the algorithm is suitable for computer-algebra systems applications. An implementation of it is written in `FORM` code to evaluate generic 1 loop and 2 loops Feynman diagrams whose fields have equal masses.

In Chapter 6, the above program is run to analytically compute all the 1 loop and 2 loops Feynman diagrams built upon the Feynman rules of ϕ^4-theory and of the single scalar field EFT, which contains a 6-point interaction vertex because its OPE is expanded up to 6-dimensional operators. The first few diagrams are also calculated by hand in parallel, to verify the veracity of the algorithm. Moreover, a symbolic loop counting method is

introduced and proved, allowing us to unambiguously determine the loop order of any n-point function contribution, even when multiple coupling constants are present.

In Chapter 7, ϕ^4-theory and the single scalar field EFT are explicitly renormalized up to 2 loops and all their beta functions and anomalous dimensions are computed at that order. In particular, ϕ^4-theory is both renormalized fully by hand and in a completely autonomous manner with additional computer-algebra codes, while the renormalization of the single scalar field EFT is done complementarily in both ways.

In the final Chapter 8, the EFT composed of two interacting scalar fields is semi-autonomously renormalized and its beta functions and anomalous dimensions are determined as well, but this time at 1 loop order. Its Green's and physical operator bases are completely characterized, and all the needed field redefinitions are found.

Chapter 2

Conventions, notations and generic formulas

2.1 Conventions

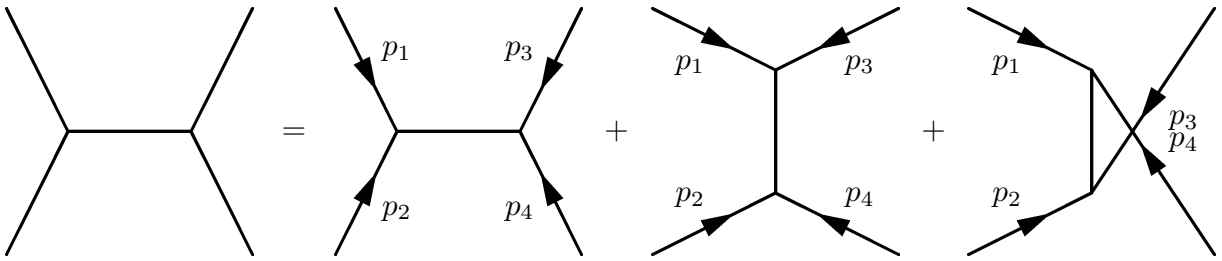
In this thesis we work in natural units $c = \hbar = 1$. For the metric tensor we use the particle physics convention $g^{\mu\nu} := \mathrm{diag}(1, -1, -1, -1)$. All considered operators will be local. In Feynman diagrams, all external momenta are assumed to be inwards. Indeed, the convention for Fourier transformations we take is

$$\begin{aligned} &\int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d k_n}{(2\pi)^d} e^{-ik_1 x_1} \cdots e^{-ik_n x_n} \tilde{f}(k_1, \ldots, k_n) = f(x_1, \ldots, x_n), \\ &\int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, e^{ik_1 x_1} \cdots e^{ik_n x_n} f(x_1, \ldots, x_n) = \tilde{f}(k_1, \ldots, k_n), \end{aligned} \tag{2.1}$$

meaning that there's no distinction between incoming and outgoing momenta.

2.2 Notations

Powers of functions are written before their argument, for example $\ln^2(z) \neq \ln(z)^2 = \ln(z^2)$. The dimension d of Dirac deltas is often omitted $\delta(x) \equiv \delta^d(x) = \delta(x^1) \cdots \delta(x^d)$, where x is a d-vector. For momenta and Feynman parameters, we use the shorthand notation $x_{i\ldots j} = x_i + \cdots + x_j$. External and loop momenta are distinguished by their choice of letter, respectively p and k. The mass dimension of a quantity q is denoted $[q]$. For a generic coupling constant C we denote C' its dimensionless version, namely $C = \mu^{[C]} C'$ where μ is called the **renormalization scale**. Graphical representations of Feynman diagrams assume the following permutation prescription:



namely all diagrams on the RHS will be summarized by a single LHS **topological diagram**. The analytical value of the LHS is the same as the RHS, and it will be written as the

analytical value of the first RHS diagram plus a symbolic $\mathcal{P}$, reminding us that we have to sum up all other (allowed) topological permutations of the diagram. The order of a list is the sum of its orders

$$\mathcal{O}(z_1, \dots, z_n) := \mathcal{O}(z_1) + \cdots + \mathcal{O}(z_n). \tag{2.2}$$

If $z_2, \dots, z_n$ are linked to z_1 by a permutation prescription we write $\mathcal{O}(z_1, \mathcal{P}) = \mathcal{O}(z_1, \dots, z_n)$. For momenta, by $\mathcal{O}(p^n)$ with $n \in \mathbb{N}$ we mean that terms of the form $(p^2)^a (k \cdot p)^b$, with $a, b \in \mathbb{N}$ and k a generic momentum, are disregarded for $2a + b \geq n$. The limiting cases $a = 0$ and $b = 0$ give an intuition for the previous condition. Hence with this new notation $\mathcal{O}(p^n) \neq \mathcal{O}\big((p^2)^{n/2}\big)$, and we prove below that

$$\int \mathrm{d}^d k \, \mathcal{O}(p^n) = \mathcal{O}\big((p^2)^{n/2}\big). \tag{2.3}$$

In general the letters d and D mean the same, but they can be differentiated if needed. Usually d is written in derived analytical results while D is written in results expected to be outputted by a computer. In loop momentum integrals, the notation $\mathrm{d}^d k$ means that k is a d-vector, while the notation $\mathrm{d}k$ means that k is a real variable.

2.3 Generic formulas

2.3.1 Recurrent series

The following three series will be widely used in this thesis [4, p.90]:

– For any $z \in \mathbb{C}$ such that $|z| < 1$ it is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \mathcal{O}\big(z^4\big); \tag{2.4}$$

– For any $z \in \mathbb{C}$ such that $|z| < 1$ it is

$$\ln(1+z) = -\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} + \mathcal{O}\big(z^4\big); \tag{2.5}$$

– For any $a, \varepsilon \in \mathbb{C}$ we have

$$a^\varepsilon = \exp(\ln(a^\varepsilon)) = \exp(\varepsilon \ln(a)) = \sum_{n=0}^{\infty} \frac{[\varepsilon \ln(a)]^n}{n!} = 1 + \varepsilon \ln(a) + \frac{\varepsilon^2}{2} \ln^2(a) + \mathcal{O}\big(\varepsilon^3\big). \tag{2.6}$$

2.3.2 Integral sign invariance

Integrations over d-vectors are invariant under a sign change of the integrated d-vector in the integrand because, by denoting k a generic d-vector,

$$\int \mathrm{d}^d k \, f(k) = \int_{\infty}^{-\infty} \mathrm{d}k^0 \dots \mathrm{d}k^d \, (-1)^d f(-k) = \int \mathrm{d}^d k \, f(-k). \tag{2.7}$$

This is a trivial result but this shows that it will also hold in Dimensional Regularization.

2.3.3 Wick rotation

A **Wick rotation** is an application of the property [5, p.172]

$$\int \mathrm{d}^4k\, f(k^2) = i \int_{\mathrm{E}} \mathrm{d}^4k\, f(-k^2), \tag{2.8}$$

where f comes from a Feynman diagram and where the E subscript indicates that k lives in Euclidean space, therefore the metric for k^2 is $\delta^{\mu\nu}$ rather than $g^{\mu\nu}$. This comes from a $k^0 = x$ identification in the property

$$\int \mathrm{d}x\, f(x) = i \int \mathrm{d}x\, f(ix), \tag{2.9}$$

which we now prove. Assume f is meromorphic over $\mathbb{C}$ and that its poles lie in either the second or the fourth quadrant: this is always the case because f comes from a Feynman diagram, and therefore it contains propagators which generate poles in $k^0 = E$ at the solutions of

$$0 = k^2 - m^2 + i\varepsilon = (E^2 - \mathbf{k}^2) - m^2 + i\varepsilon, \tag{2.10}$$

which are indeed found to be [6, p.114]

$$E^*_\pm = \pm\sqrt{\mathbf{k}^2 + m^2 - i\varepsilon} \approx \pm\left(\sqrt{\mathbf{k}^2 + m^2} - \frac{i\varepsilon}{2\sqrt{\mathbf{k}^2 + m^2}}\right). \tag{2.11}$$

We can then safely apply Cauchy's theorem

$$0 = \int_{C_\pm} \mathrm{d}z\, f(z), \tag{2.12}$$

where $C_\pm$ are the contours of figure 2.1 and $a \in \mathbb{R}$. For the C_+ path, parameterize the real axis with $z(\lambda) = \lambda$, the arc with $z(\lambda) = ae^{i\lambda}$ and the imaginary axis with $z(\lambda) = i\lambda$ to get

$$\int_0^a \mathrm{d}x\, f(x) = i \int_0^a \mathrm{d}x\, f(ix). \tag{2.13}$$

The arc contribution is null given that f comes from a Feynman diagram. Repeating the procedure for the C_- path gives

$$\int_{-a}^0 \mathrm{d}x\, f(x) = i \int_{-a}^0 \mathrm{d}x\, f(ix). \tag{2.14}$$

These two integrals can be summed and finally, by letting $a \to \infty$, we show the proposition.

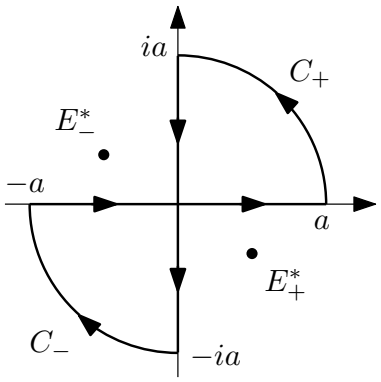


Figure 2.1: Complex plane with the poles of f in k^0 and the contours $C_\pm$ avoiding them.

2.3.4 Integral of an $\mathcal{O}(p^n)$

We prove the (2.3) result. Define the new d-momentum $q^\mu := p^\mu$ for $\mu > 0$ and $iq^0 := p^0$, then if f is a function such that the whole integral converges we can compute

$$\begin{aligned}\int \mathrm{d}^d k\, \mathcal{O}(p^n) &= \int \mathrm{d}^d k\, (p^2)^a (k\cdot p)^b f(k^2) = i(p^2)^a \int_{\mathrm{E}} \mathrm{d}^d k\, (-k\cdot q)^b f(-k^2) \\ &= i(p^2)^a \int_{\mathrm{E}} \mathrm{d}^d k\, (-Rk\cdot Rq)^b f(-k^2) = i(p^2)^a \int_{\mathrm{E}} \mathrm{d}^d k\, \big(-(Rk)^0 \|q\|_{\mathrm{E}}\big)^b f(-k^2) \\ &= (p^2)^a (p^2)^{b/2} \int \mathrm{d}^d k\, (ik^0)^b f(k^2) = \mathcal{O}\big((p^2)^{n/2}\big),\end{aligned} \tag{2.15}$$

where $R \in \mathrm{SO}(d)$ is an orthogonal transformation that rotates the q vector to the first axis $(Rq)^\mu = \delta^{0\mu}\|q\|_{\mathrm{E}}$, the norm $\|q\|_{\mathrm{E}}^2 = q^2$ is computed with the $\delta^{\mu\nu}$ metric and in the last equality the integral converges by construction.

2.3.5 Integral representations

Define the following formal notations

$$\mathrm{d}\mathbf{x} := \mathrm{d}x_1 \cdots \mathrm{d}x_n \left(x_1^{\nu_1 - 1} \cdots x_n^{\nu_n - 1}\right) \qquad \text{and} \qquad \int_{\nu_1 \ldots \nu_n} \mathrm{D}x := \frac{(-1)^{\nu_1} \cdots (-1)^{\nu_n}}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_0^\infty \mathrm{d}\mathbf{x}\,. \tag{2.16}$$

Then for any $A_i \in \mathbb{R}$ and $\nu_i \in \mathbb{N}$ it can be shown that [7, p.42]

$$\frac{1}{A_1^{\nu_1} \cdots A_n^{\nu_n}} = \frac{\Gamma(\nu_1 + \cdots + \nu_n)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_0^1 \mathrm{d}\mathbf{x}\, \frac{\delta(1 - \sum_i x_i)}{(\sum_i x_i A_i)^{\sum_i \nu_i}}, \tag{2.17}$$

which is called the **Feynman representation**, and that if we know that all $A_i < 0$ (it usually happens after a Wick rotation) we have [7, p.37]

$$\frac{1}{A_1^{\nu_1} \cdots A_n^{\nu_n}} = \frac{(-1)^{\nu_1} \cdots (-1)^{\nu_n}}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_0^\infty \mathrm{d}\mathbf{x} \exp\left(\sum_i x_i A_i\right) = \int_{\nu_1 \ldots \nu_n} \mathrm{D}x\, \exp\left(\sum_i x_i A_i\right), \tag{2.18}$$

which is called the **Schwinger representation**.

2.4 Generating functionals

In this section, we give a quick refresher on generating functionals and we introduce notations and conventions for their generated functions.

Consider a collection of N scalar fields $\vec{\phi} := (\phi_1, \ldots, \phi_N)$. Given a theory specified by a Lagrangian (density) $\mathcal{L}[\vec{\phi}]$, one defines the action functional as

$$S[\vec{\phi}] := \int \mathrm{d}^d x\, \mathcal{L}[\vec{\phi}(x)]. \tag{2.19}$$

By taking the $\vec{\phi}$-functional derivative of the action, the configurations of fields that make it stationary are those satisfying the Euler-Lagrange **equations of motion** (**EOMs**)

$$\vec{\mathbf{0}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\phi})}\right) - \frac{\partial \mathcal{L}}{\partial \vec{\phi}} \tag{2.20}$$

and they are called classical fields. The action can be cast inside a path integral together with sources $\vec{\mathbf{J}} := (J_1, \dots, J_n)$, associated to their respective $\vec{\phi}$ fields, to define a new quantity called the generating functional [8, p.24]

$$Z[\vec{\mathbf{J}}] := \frac{\int \mathrm{D}\vec{\phi}\, \exp\left[i\left(S[\vec{\phi}] + \int \mathrm{d}^d x\, \vec{\mathbf{J}}(x)\cdot\vec{\phi}(x)\right)\right]}{\int \mathrm{D}\vec{\phi}\, \exp\left(iS[\vec{\phi}]\right)}. \tag{2.21}$$

Here we made the choice to normalize it right away in its definition because that way one has $Z[\vec{\mathbf{0}}] = 1$ without further ado, however in this thesis we'll often consider its unrenormalized version. With that, one can additionally define the connected diagrams generating functional to be that quantity $W[\vec{\mathbf{J}}]$ such that [8, p.26]

$$Z[\vec{\mathbf{J}}] = e^{iW[\vec{\mathbf{J}}]}, \tag{2.22}$$

and the quantum effective action functional [8, p.66]

$$\Gamma[\vec{\phi}] := W[\vec{\mathbf{J}}] - \int \mathrm{d}^d x \left(\vec{\mathbf{J}}(x)\cdot\vec{\phi}(x)\right). \tag{2.23}$$

2.4.1 Obtaining n-point functions

The point of defining all these generating functionals is that they allow us, as their name suggests, to generate correlation functions, or Green's functions, namely vacuum expectation values of time-ordered fields. What distinguishes a generating functional from another is what kind of diagrams are summed inside its generated Green's functions: for $(i_1, \dots, i_n) \in \{1, \dots, N\}^n$ one has that

- $Z[\vec{\mathbf{J}}]$ generates the so-called **n-point Green's functions** [8, p.17]

$$\langle \phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle := \frac{1}{i^n}\,\frac{\delta^n Z[\vec{\mathbf{J}}]}{\delta J_{i_1}(x_1)\cdots\delta J_{i_n}(x_n)}\bigg|_{\vec{\mathbf{J}}=\vec{\mathbf{0}}} = \frac{\int \mathrm{D}\vec{\phi}\,[\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)]e^{iS[\vec{\phi}]}}{\int \mathrm{D}\vec{\phi}\, e^{iS[\vec{\phi}]}}, \tag{2.24}$$

  which contain all possible diagrams without vacuum bubbles with n external legs;

- $W[\vec{\mathbf{J}}]$ generates **n-point connected Green's functions** [8, p.26]

$$G^{(n)}_{\phi_{i_1}\dots\phi_{i_n}}(x_1, \dots, x_n) = \langle \phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle_{\text{Connected}} := \frac{1}{i^{n-1}}\,\frac{\delta^n W[\vec{\mathbf{J}}]}{\delta J_{i_1}(x_1)\cdots\delta J_{i_n}(x_n)}\bigg|_{\vec{\mathbf{J}}=\vec{\mathbf{0}}}, \tag{2.25}$$

  which contain only connected diagrams with n external legs;

- $\Gamma[\vec{\phi}]$ generates **1PI n-point Green's functions** [8, p.73]

$$\Gamma^{(n)}_{\phi_{i_1}\dots\phi_{i_n}}(x_1, \dots, x_n) := \frac{\delta^n \Gamma[\vec{\phi}]}{\delta\phi_{i_1}(x_1)\cdots\delta\phi_{i_n}(x_n)}, \tag{2.26}$$

  which contain only one-particle irreducible (1PI) diagrams with n external **amputated legs**, meaning that no full propagator is provided for any external leg, as in figure 2.2.

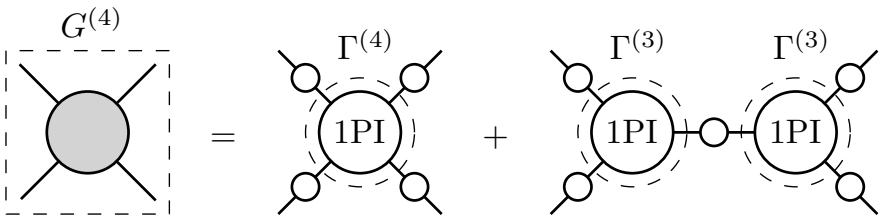


Figure 2.2: Relationship between connected and 1PI Green's functions. In this $n = 4$ example, $\Gamma^{(4)}$ and $\Gamma^{(3)}$ contain 1PI diagrams, while $G^{(4)}$ is obtained by providing full propagators to each of the external (with respect to the 1PI bubbles) legs. The $\mathcal{P}$ prescription applies.

With the above definitions, one can give an expression of the generating functionals in terms of their generated functions:

$$\begin{aligned}
Z[\vec{\mathbf{J}}] &= 1 + \sum_{n=1}^{\infty} \left(\sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} \right) \frac{1}{n!} \prod_{j=1}^{n} \left(i \int \mathrm{d}^d x_j \, J_{i_j}(x_j) \right) \langle \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) \rangle \,, \\
W[\vec{\mathbf{J}}] &= 1 + \sum_{n=1}^{\infty} \left(\sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} \right) \frac{1}{n!} \prod_{j=1}^{n} \left(i \int \mathrm{d}^d x_j \, J_{i_j}(x_j) \right) G^{(n)}_{\phi_{i_1} \dots \phi_{i_n}}(x_1, \dots, x_n), \\
\Gamma[\vec{\phi}] &= 1 + \sum_{n=1}^{\infty} \left(\sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} \right) \frac{1}{n!} \prod_{j=1}^{n} \left(\int \mathrm{d}^d x_j \, \phi_{i_j}(x_j) \right) \Gamma^{(n)}_{\phi_{i_1} \dots \phi_{i_n}}(x_1, \dots, x_n),
\end{aligned} \tag{2.27}$$

where a symmetry factor of $1/(n!)$ is provided to compensate for the over-counting of the Green's functions whose only difference is the order of their fields (time-ordering doesn't care about any other ordering). The $N = 1$ case, setting $\phi_1 = \phi$ and $J_1 = J$, gives us back the familiar expansions [8, p.74]

$$\begin{aligned}
Z[J] &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \left(i \int \mathrm{d}^d x_j \, J(x_j) \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle \,, \\
W[J] &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \left(i \int \mathrm{d}^d x_j \, J(x_j) \right) G^{(n)}(x_1, \dots, x_n), \\
\Gamma[\phi] &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \left(\int \mathrm{d}^d x_j \, \phi(x_j) \right) \Gamma^{(n)}(x_1, \dots, x_n),
\end{aligned} \tag{2.28}$$

where from now on we'll refrain from writing the fields below the Green's functions if they are all the same or if they can be univocally determined. A series of other conventions and abuses of language/notation will be assumed through this thesis:

- n-point Green's functions won't be considered;
- n-point connected Green's functions will be called **(regular) n-point functions**;
- 1PI n-point Green's functions will be called **1PI n-point functions**;
- Regular and 1PI n-point functions Fourier-transformed in momentum space will be denoted with the same symbols but with different arguments

$$\begin{aligned}
\tilde{G}^{(n)}(p_1, \dots, p_n) &\equiv G^{(n)}(p_1, \dots, p_n) \neq G^{(n)}(x_1, \dots, x_n), \\
\tilde{\Gamma}^{(n)}(p_1, \dots, p_n) &\equiv \Gamma^{(n)}(p_1, \dots, p_n) \neq \Gamma^{(n)}(x_1, \dots, x_n);
\end{aligned} \tag{2.29}$$

– If a relationship holds for both types of arguments, they will be dropped altogether

$$\begin{aligned} G^{(n)} &\equiv G^{(n)}(x_1, \dots, x_n) \equiv G^{(n)}(p_1, \dots, p_n), \\ \Gamma^{(n)} &\equiv \Gamma^{(n)}(x_1, \dots, x_n) \equiv \Gamma^{(n)}(p_1, \dots, p_n); \end{aligned} \tag{2.30}$$

– All n-point functions will omit the canonical $(2\pi)^4\delta(p_1+\cdots+p_n)$ factor ensuring global 4-momentum conservation of processes.

2.4.2 The 1PI 2-point function

For $n > 2$, all 1PI n-point functions are obtained by summing over all 1PI diagrams with n external legs. However, the $n = 2$ case doesn't follow this simple rule. To see it, denote by $\Delta(p)$ the free propagator in momentum space and by $\Sigma(p)$ the sum of all 1PI 2-point diagrams. We want to show that $\Gamma^{(2)} \neq \Sigma(p)$. To do so, we compute the full propagator [8, p.70]

$$G^{(2)}(p) = \Delta(p) \sum_{n=0}^{\infty} \Big[\Sigma(p)\Delta(p)\Big]^n = \frac{\Delta(p)}{1 - \Sigma(p)\Delta(p)} = \frac{1}{\Delta^{-1}(p) - \Sigma(p)} \tag{2.31}$$

and, using the fact that $\Gamma^{(2)}$ is the inverse of $G^{(2)}$ [8, p.72], one obtains

$$\Gamma^{(2)}(p) = \Delta^{-1}(p) - \Sigma(p). \tag{2.32}$$

So, in reality, the 1PI 2-point function is given by the inverse of the free propagator minus all 1PI diagrams. Notice that, however, we could have instead obtained

$$\Gamma^{(2)}(p) = \Sigma(p) - \Delta^{-1}(p) \tag{2.33}$$

had we defined the generating functionals such that $\Gamma^{(2)}$ would have been the negative inverse of $G^{(2)}$. In this latter case, $\Gamma^{(2)}$ would have been indeed the sum of all 1PI diagrams, minus the inverse of the free propagator. Ultimately, it will become clear that this global sign difference won't affect renormalization, as long as the relative sign difference between the two terms involved is -1. As a final remark notice that, thanks to the conservation of momentum $p_{12} = 0$, all of the above quantities only have one momentum argument $p = p_1$.

Chapter 3

Regularization and renormalization

3.1 Regularization

3.1.1 Motivation

Given a set of Feynman rules, they can be applied to determine the analytical form of a diagram's amplitude. However, there's no guarantee whatsoever about the convergence of said amplitude: the obtained analytical expressions can be ill-defined and hence diverge [7, p.xi], and as result we often encounter situations of the kind

$$\int \mathrm{d}^4k\, f(k^2) > \infty, \tag{3.1}$$

where f comes from a Feynman diagram. Nevertheless, one might still be interested to give a somewhat useful meaning to the divergent amplitudes. Indeed, if one finds out that the total divergence of an amplitude is driven by the divergence of one of its parts, then there's an hope to isolate the latter and therefore to be able to extract remaining finite, meaningful, parts. The procedure of **regularization**, given a chosen **regularization scheme**, consists of exactly quantifying this divergence such that it can be subtracted from the original amplitude.

3.1.2 Dimensional regularization

In this thesis we work in the **Dimensional Regularization** scheme (**DR**) because, as it shall become obvious in this presentation section, it doesn't break Lorentz invariance.

Consider the diverging integral in the above motivational section and define instead

$$F(D) := \int \mathrm{d}^Dk\, f(k^2) \tag{3.2}$$

with $D \in \mathbb{N}^*$. After a Wick rotation (2.8), because we end up in a Euclidean space and because D is an integer, we can change variables to the so-called **hyperspherical coordinates** [8, p.110], which generalize the common spherical coordinates, giving us [9]

$$F(D) = i\int_{\mathrm{E}} \mathrm{d}^Dk\, f(-k^2) = i\Omega_D \int_0^\infty \mathrm{d}k\, k^{D-1} f(-k^2), \tag{3.3}$$

where Ω_D is the surface area of the D-dimensional hypersphere [9]

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{3.4}$$

We observe that $F(4) > \infty$, as it should, but we also notice that since f comes from a Feynman diagram then $F(D) < \infty$ for some other values of D in some subset $D \in U \subset \mathbb{R}$, even for non-integer ones. We therefore interpret $F(4) > \infty$ as a pole at $d = 4$ of an analytically continued (AC) function in the complex plane [9]

$$G(d) := \int \mathrm{d}^d k\, f(k^2) := \underset{\mathrm{d}}{\mathrm{AC}} \left[i\Omega_d \int_0^\infty \mathrm{d}k\, k^{d-1} f(-k^2) \right] \tag{3.5}$$

with now $d \in V \subset \mathbb{C}$ a complex dimension in some bigger subset $V \supset U$. Notice that the AC operation in d is well-defined because the analytical continuation of a function is unique. For integer d it must be $G(d) = F(d)$, and the analytical continuation of Ω_d is given by the analytical continuation of the Gamma function

$$\Gamma(z) := \underset{\mathrm{z}}{\mathrm{AC}} \left[\int_0^\infty \mathrm{d}x\, x^{z-1} e^{-x} \right] \tag{3.6}$$

which, in its analytically extended form out of its original $\mathrm{Re}(z) > 0$ domain, has poles in $z \in \mathbb{Z}_-$ and is therefore meromorphic [10]. The function has the notable $\Gamma(z+1) = z\Gamma(z)$ property and it has a $z = 0$ pole where it assumes a Laurent series expansion [10]

$$\Gamma(z) = \frac{1}{z} - \gamma + z\gamma' + \mathcal{O}\big(z^2\big), \tag{3.7}$$

with $\gamma = \gamma_E$ the Euler-Mascheroni constant and

$$\gamma' := \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right). \tag{3.8}$$

In general we are interested in dimensions $\mathrm{Re}(d) > 0$, therefore only this $z = 0$ pole will be of interest.

We'll say that $F(4)$ has been dimensionally regularized into $G(d)$ and, when writing down integrals of the form (3.5), we'll omit the AC operation which will be assumed to be implicitly there. In this picture, it is now clear that the $d = 4$ pole of G corresponds to the divergence of $F(4) = G(4)$. In general, G might even have additional poles: in order to exactly quantify the divergences on each of them, we'll describe the asymptotic behaviour of G in a narrow neighbourhood of said poles. In particular, for the $d = 4$ pole, we restrain ourselves on the real line and set

$$d = 4 - 2\varepsilon \tag{3.9}$$

with $\varepsilon > 0$ small. If all the additional poles also lie on the real line, ε can also be used to probe their neighbourhoods, but in that case ε won't be small. In any case, once we analytically extend amplitudes, we can't touch their poles anymore, that is, we can't formally consider values of d that made the integral diverge anymore.

3.1.3 Deriving the general tadpole

As a first application, we apply DR to regularize the **general tadpole integral**

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(k^2)^a}{(k^2 - \sigma^2)^b}, \tag{3.10}$$

with $a, b \in \mathbb{R}$. It is clear that for some values of a and b the integral diverges, therefore we go into $d \in \mathbb{C}$ dimensions and compute the analytically continued integral [11, p.827]

$$\begin{aligned}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k^2)^a}{(k^2-\sigma^2)^b} &= i(-1)^{a+b} \int_{\mathrm{E}} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k^2)^a}{(k^2+\sigma^2)^b} \\ &= \frac{i\Omega_d}{(2\pi)^d} \int \mathrm{d}k \, \frac{(-1)^{a+b} k^{2a+d-1}}{(k^2+\sigma^2)^b} \\ &= \frac{i\Omega_d}{(2\pi)^d} \frac{(-1)^{a+b}}{2} \sigma^{2a-2b+d} \int_0^1 \mathrm{d}x \, x^{a+d/2-1} (1-x)^{b-a-d/2-1} \\ &= \frac{i(-1)^{a+b}}{(4\pi)^{d/2}} \sigma^{2a-2b+d} \frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(b-a-\frac{d}{2}\right)}{\Gamma(b)\Gamma\left(\frac{d}{2}\right)},\end{aligned} \tag{3.11}$$

where in the first line we applied a Wick rotation (2.8), in the second line we went into hyperspherical coordinates, in the third line we changed variables to [6, p.158]

$$\begin{cases} k^2 = \dfrac{x}{1-x}\sigma^2 \\ \mathrm{d}k = \dfrac{\sigma^2}{2k(1-x)^2}\,\mathrm{d}x \end{cases} \tag{3.12}$$

and in the fourth line we recognized the Beta function

$$B(\alpha,\beta) := \underset{\alpha,\beta}{\mathrm{AC}} \left[\int_0^1 \mathrm{d}x \, x^{\alpha-1}(1-x)^{\beta-1} \right] = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{3.13}$$

In particular, (3.11) implies that the usual tadpole integral is [8, p.111]

$$T_\nu^D := \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2-\sigma^2)^\nu} = C(\nu)\left(\sigma^2\right)^{\frac{D}{2}-\nu} = C(\nu)\sigma^{D-2\nu}, \tag{3.14}$$

with the constant

$$C(\nu) := \frac{i(-1)^\nu}{(4\pi)^{D/2}} \frac{\Gamma(\nu-\frac{D}{2})}{\Gamma(\nu)}. \tag{3.15}$$

Its divergence for $\nu = 1$ around the pole $d = 4$ becomes manifest if we set $d = 4 - 2\varepsilon$ and expand the Gamma function as in (3.7) up to first order [8, p.111]:

$$\begin{aligned} T_1^d &= \frac{i\sigma^2}{(4\pi)^2}\left(\frac{4\pi}{\sigma^2}\right)^\varepsilon \left[\frac{1}{\varepsilon} + 1 - \gamma + \mathcal{O}(\varepsilon)\right] = \frac{i\sigma^2}{(4\pi)^2}\left(\frac{4\pi}{\sigma^2}\right)^\varepsilon \left[\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right]\left(1 - \varepsilon\gamma + \mathcal{O}\left(\varepsilon^2\right)\right) \\ &= \frac{i\sigma^2}{(4\pi)^2}\left(\frac{4\pi e^{-\gamma}}{\sigma^2}\right)^\varepsilon \left[\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right] = i\sigma^2 N\left[\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right], \end{aligned} \tag{3.16}$$

where N is the normalization constant

$$N := \frac{1}{Q}\left(\frac{4\pi e^{-\gamma}}{\sigma^2}\right)^\varepsilon \quad \text{with} \quad Q := (4\pi)^2. \tag{3.17}$$

Indeed, as $\varepsilon \to 0$, T_1^d clearly diverges. The ε-pole in T_1^d is its diverging part, while everything else is its finite part. The $\mathcal{O}(\varepsilon)$ terms are conventionally excluded from the finite part because they vanish in the $\varepsilon \to 0$ limit, however they might become part of it should they get multiplied with an $1/\varepsilon$-pole: in that case an expansion up to $\mathcal{O}(\varepsilon^2)$ is needed to fully determine the finite part. Lastly, if one defines the **$\overline{\mathrm{MS}}$/modified scale**

$$\bar{\mu}^2 := (4\pi e^{-\gamma})\mu^2 \tag{3.18}$$

and the modified normalization constant

$$\bar{N} := \mu^{2\varepsilon} N = \frac{1}{Q}\left(\frac{4\pi e^{-\gamma}\mu^2}{\sigma^2}\right)^{\varepsilon} = \frac{1}{Q}\left(\frac{\bar{\mu}^2}{\sigma^2}\right)^{\varepsilon}, \tag{3.19}$$

then the usual tadpole T_1^D multiplied by the renormalization scale $\mu^{2\varepsilon}$ becomes

$$\mu^{2\varepsilon} T_1^D = i\sigma^2 \bar{N}\left[\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right]. \tag{3.20}$$

3.1.4 Deriving the sunset

As a second application of DR, we regularize the **sunset integral**

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{k^2-\sigma^2}\frac{1}{l^2-\sigma^2}\frac{1}{(k-l)^2-\sigma^2}. \tag{3.21}$$

For reasons that will become clear in Chapter 5, its dimensional regularized version will be called I_{111}^D. Its evaluation is much harder than (3.11) and it requires a few non-trivial tricks:

$$\begin{aligned}
I_{111}^D &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{k^2-\sigma^2}\frac{1}{l^2-\sigma^2}\frac{1}{(k-l)^2-\sigma^2} \\
&= \Gamma(3)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{\left(x_{13}k^2 + x_{23}l^2 - \sigma^2 - 2x_3(k\cdot l)\right)^3} \\
&= \Gamma(3)\int_0^1 \mathrm{d}\mathbf{x}\int \frac{\mathrm{d}^d k'}{(2\pi)^d}\frac{\mathrm{d}^d l'}{(2\pi)^d}\frac{\delta(1-x_{123})}{\left[x_{13}k'^2 + \left(-\frac{x_3^2}{x_{13}} + x_{23}\right)l'^2 - \sigma^2\right]^3} \\
&= \Gamma(3)\int_0^1 \mathrm{d}\mathbf{x}\, a^{-3}\int \frac{\mathrm{d}^d k'}{(2\pi)^d}\frac{\mathrm{d}^d l'}{(2\pi)^d}\frac{\delta(1-x_{123})}{\left(k'^2 - \frac{\sigma^2 - b l'^2}{a}\right)^3} \\
&= \Gamma(3)\int_0^1 \mathrm{d}\mathbf{x}\, a^{-3}\int \frac{\mathrm{d}^d l'}{(2\pi)^d} C(3)\left(\frac{\sigma^2 - b l'^2}{a}\right)^{\frac{d}{2}-3}\delta(1-x_{123}) \\
&= \Gamma(3)C(3)(-1)^{\varepsilon+1}\int_0^1 \mathrm{d}\mathbf{x}\, a^{\varepsilon-2}b^{-1-\varepsilon}\int \frac{\mathrm{d}^d l'}{(2\pi)^d}\frac{\delta(1-x_{123})}{\left(l'^2 - \frac{\sigma^2}{b}\right)^{\varepsilon+1}} \\
&= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\sigma^{2-4\varepsilon}\int_0^1 \mathrm{d}\mathbf{x}\,(ab)^{\varepsilon-2}\delta(1-x_{123}),
\end{aligned} \tag{3.22}$$

where in the second line we go into Feynman representation (with unity power indices) and we immediately simplify $x_{123} = 1$ because of the delta function, in the third line we change variables to

$$\begin{cases} k = k' + \dfrac{x_3}{x_{13}} l' \\ l = l' \end{cases} \tag{3.23}$$

in order to diagonalize the denominator, in the fourth line we define and substitute the auxiliary variables

$$\begin{cases} a = x_{13} \\ b = -\dfrac{x_3^2}{x_{13}} + x_{23} \end{cases} \tag{3.24}$$

so that in the fifth and seventh line (3.14) can be applied [12, p.48-49]. All that is left to determine is an integration over $\mathbf{x}$ that can be computed as [12, p.49-50]

$$\begin{aligned}\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{(ab)^{2-\varepsilon}} &= \int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_1-x_2-x_3)}{(x_1x_2+x_2x_3+x_1x_3)^{2-\varepsilon}}\\ &= 6\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_1-x_2-x_3)}{(x_1x_2+x_2x_3+x_1x_3)^{2-\varepsilon}}\theta(x_1>x_2)\theta(x_2>x_3)\\ &= 6\int_0^1 \mathrm{d}\mathbf{u}\,\left(u_1^2u_2\right)^{\varepsilon-1}\frac{\delta[1-u_1(1+u_2(1+u_3))]}{(1+u_3(1+u_2))^{2-\varepsilon}}\\ &= 6\int_0^1 \mathrm{d}u_2\,\frac{1}{u_2^{1-\varepsilon}}\int_0^1 \mathrm{d}u_3\,\frac{[1+u_2(1+u_3)]^{1-2\varepsilon}}{[1+u_3(1+u_2)]^{2-\varepsilon}}\\ &= 6\int_0^1 \mathrm{d}u_2\,\frac{1}{u_2^{1-\varepsilon}}\Big\{[F(u_2)-F(0)]+F(0)\Big\}.\end{aligned} \tag{3.25}$$

We now explain line by line the above calculation. In the second line, we multiplied the integrand by the tautology

$$1=\sum_{\alpha\in S_3}\theta(x_{\alpha(1)}>x_{\alpha(2)}>x_{\alpha(3)})=\sum_{\alpha\in S_3}\theta(x_{\alpha(1)}>x_{\alpha(2)})\theta(x_{\alpha(2)}>x_{\alpha(3)}), \tag{3.26}$$

where S_3 is the permutation group of degree 3, α is a permutation map $\alpha:\{1,2,3\}\to\{1,2,3\}$ and $\theta:\{\text{Propositions}\}\to\{0,1\}$ is the Heaviside step function that returns 0 if the proposition if false and 1 if it is true. Since the integral is invariant under index permutation of the Feynman parameters, the sum transforms into a multiplicative factor of $|S_3|=3!=6$. In the third line, the two Heaviside step functions are implemented as a change of variables

$$\mathbf{x}(\mathbf{u})=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}u_1\\u_1u_2\\u_1u_2u_3\end{pmatrix}, \tag{3.27}$$

with $\mathbf{u}=(u_1,u_2,u_3)$ and still $u_i\in[0;1]$, implying a Jacobian determinant of

$$|\det(J_{\mathbf{x}})|=\left|\det\begin{pmatrix}1&0&0\\u_2&u_1&0\\u_2u_3&u_1u_3&u_1u_2\end{pmatrix}\right|=\left|u_1^2u_2\right|=u_1^2u_2. \tag{3.28}$$

In the fourth line, we used the scaling property of the delta function while getting rid of it in the u_1 integration. More in detail, we applied the property

$$\begin{aligned}\int_0^1 \mathrm{d}u_1\,\delta(1-u_1\alpha(u_2,u_3))f(\mathbf{u}) &= \int_{-\infty}^{\infty}\mathrm{d}u_1\,\delta(1-u_1\alpha(u_2,u_3))f(\mathbf{u})\\ &= \int_{-\infty}^{\infty}\mathrm{d}u_1'\,\frac{1}{\alpha(u_2,u_3)}\delta(1-u_1')f\left(\frac{u_1'}{\alpha(u_2,u_3)},u_2,u_3\right)\\ &= \frac{1}{\alpha(u_2,u_3)}f\left(\frac{1}{\alpha(u_2,u_3)},u_2,u_3\right),\end{aligned} \tag{3.29}$$

where $f:\mathbb{R}^3\to\mathbb{R}$ is a generic function and $\alpha:[0;1]^2\to\mathbb{R}$ is defined as $\alpha(u_2,u_3):=1+u_2(1+u_3)$. Because $u_2,u_3\in[0;1]$ then $\alpha\in[1;3]$, meaning that $1/\alpha\in[1/3;1]\subset[0;1]$

and therefore the u_1-integration can be extended from $[0;1]$ to the real line, with the change of variables $u_1' = u_1\alpha$. In the fifth line, we simply defined

$$F(u_2) := \int_0^1 \mathrm{d}u_3 \, \frac{\left[1+u_2(1+u_3)\right]^{1-2\varepsilon}}{\left[1+u_3(1+u_2)\right]^{2-\varepsilon}}. \tag{3.30}$$

This is done because, since for $\varepsilon \neq 1$ it is

$$F(0) = \int_0^1 \mathrm{d}u_3 \, \frac{1}{(1+u_3)^{2-\varepsilon}} = \frac{1-2^{\varepsilon-1}}{1-\varepsilon} \xrightarrow{\varepsilon\to 0} \frac{1}{2}, \tag{3.31}$$

then it can be analytically computed through `Mathematica` that

$$\int_0^1 \mathrm{d}u_2 \, \frac{F(u_2)-F(0)}{u_2^{1-\varepsilon}}\bigg|_{\varepsilon=0} = \frac{\ln(2)}{2} < \infty, \tag{3.32}$$

which is therefore also finite for any value of ε up to $\mathcal{O}(\varepsilon)$ additional terms. This means that the first part of the integral constitutes its finite part, while it is the second part that provides its diverging part

$$\begin{aligned} \int_0^1 \mathrm{d}\mathbf{x} \, \frac{\delta(1-x_{123})}{(ab)^{2-\varepsilon}} &= 6\left[\left(\frac{\ln(2)}{2}+\mathcal{O}(\varepsilon)\right) + \int_0^1 \mathrm{d}u_2 \, \frac{F(0)}{u_2^{1-\varepsilon}}\right] = 3\ln(2) + \frac{6F(0)}{\varepsilon} + \mathcal{O}(\varepsilon) \\ &= 3\ln(2) + \frac{3}{\varepsilon}\Big(1+\varepsilon\big[1-\ln(2)\big]\Big) + \mathcal{O}(\varepsilon) = \frac{3}{\varepsilon} + 3 + \mathcal{O}(\varepsilon), \end{aligned} \tag{3.33}$$

where in the third equality $F(0)$ was expanded in ε thanks to (2.6) and (2.4). This was to be expected because the integrand of $F(u_2)$ is integrable in the $[0;1]^2 \subset \mathbb{R}^2$ region, hence singularities can only arise through the u_2-integration. Having determined the value of this integral, we can return to the original one and expand everything in ε to finally get [12, p.50] [13, p.107]

$$\begin{aligned} I^D_{111} &= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\sigma^{2-4\varepsilon}\left(\frac{3}{\varepsilon}+3+\mathcal{O}(\varepsilon)\right) \\ &= \frac{\sigma^{2-4\varepsilon}}{(4\pi)^d}\Gamma(2\varepsilon-1)\left(\frac{3}{\varepsilon}+3+\mathcal{O}(\varepsilon)\right) \\ &= \frac{-\sigma^2}{(4\pi)^4}\left(\frac{4\pi}{\sigma^2}\right)^{2\varepsilon}\frac{\Gamma(2\varepsilon)}{1-2\varepsilon}\left(\frac{3}{\varepsilon}+3+\mathcal{O}(\varepsilon)\right) \\ &= -\frac{3}{2}\frac{\sigma^2}{(4\pi)^4}\left(\frac{4\pi}{\sigma^2}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(3-2\gamma\Big)+\mathcal{O}\big(\varepsilon^0\big)\right] \\ &= -\frac{3}{2}\frac{\sigma^2}{(4\pi)^4}\left(\frac{4\pi}{\sigma^2}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon^2}+\frac{3}{\varepsilon}+\mathcal{O}\big(\varepsilon^0\big)\right]\Big(1-2\varepsilon\gamma+\mathcal{O}\big(\varepsilon^2\big)\Big) \\ &= -\frac{3}{2}\frac{\sigma^2}{(4\pi)^4}\left(\frac{4\pi e^{-\gamma}}{\sigma^2}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon^2}+\frac{3}{\varepsilon}+\mathcal{O}\big(\varepsilon^0\big)\right], \end{aligned} \tag{3.34}$$

implying

$$\mu^{4\varepsilon} I^D_{111} = -\frac{3}{2}\sigma^2\bar{N}^2\left[\frac{1}{\varepsilon^2}+\frac{3}{\varepsilon}+\mathcal{O}\big(\varepsilon^0\big)\right]. \tag{3.35}$$

3.1.5 Deriving the semi-general sunset

As a third application of DR, we regularize the **semi-general sunset integral**

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{k^2-\rho^2}\frac{1}{l^2-\sigma^2}\frac{1}{(k-l)^2-\sigma^2}, \tag{3.36}$$

called that way because there are two masses $\rho \neq \sigma$ for three distinct propagators. Again, for reasons that will become clear in Chapter 5, its dimensional regularized version will be called H^D_{111}. Its evaluation is analogous to the procedure we just saw for I^D_{111}, with just a few adjustments needed, therefore the former was put in Appendix A. It is found that

$$\mu^{4\varepsilon}H^D_{111} = -\frac{1}{2}\bar{N}^2\left\{\frac{1}{\varepsilon^2}\left(\rho^2+2\sigma^2\right)+\frac{1}{\varepsilon}\left[3\rho^2+6\sigma^2-2\rho^2\ln\left(\frac{\rho}{\sigma}\right)^2\right]+\mathcal{O}\left(\varepsilon^0\right)\right\}. \tag{3.37}$$

This result agrees with the ε-expansion given by [14]. As a check, we can set $\rho = \sigma$ to readily recover the sunset integral (3.35).

3.1.6 Deriving the general sunset

As a final application of DR, we regularize the **general sunset integral**

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{k^2-\rho^2}\frac{1}{l^2-\sigma^2}\frac{1}{(k-l)^2-\tau^2}, \tag{3.38}$$

called that way because all masses are assumed to be different $\rho \neq \sigma \neq \tau \neq \rho$. Once more, for reasons that will become clear in Chapter 5, its dimensional regularized version will be called K^D_{111}. Its evaluation uses exactly the same machinery previously employed for H^D_{111}, therefore it was put in Appendix B. There, it is found that

$$\begin{aligned}\mu^{4\varepsilon}K^D_{111} = -\frac{1}{2}\bar{N}^2\Bigg\{&\frac{1}{\varepsilon^2}\left(\rho^2+\sigma^2+\tau^2\right)\\&+\frac{1}{\varepsilon}\left[3\rho^2+3\sigma^2+3\tau^2-2\rho^2\ln\left(\frac{\rho}{\sigma}\right)^2-2\tau^2\ln\left(\frac{\tau}{\sigma}\right)^2\right]+\mathcal{O}\left(\varepsilon^0\right)\Bigg\}.\end{aligned} \tag{3.39}$$

Again, the result agrees with the ε-expansion given in [14]. As a check, we can set $\tau = \sigma$ to recover (3.37) and $\tau = \rho = \sigma$ to recover (3.35).

3.2 Renormalization

3.2.1 Motivation

In the previous section, thanks to DR, we found a way to quantify the divergences of Feynman diagrams. This made it possible to identify their finite and diverging parts, meaning that the two can be separated and, by removing the latter, finite amplitudes can be extracted. The procedure of **renormalization**, given a certain **renormalization scheme** specifying a systematic way to remove said divergences, allows us to do exactly that. There's a certain freedom in the choice of the renormalization scheme because one can decide, in addition to the diverging parts, to also remove bits of the finite parts such to obtain other desired results, like a specific full propagator pole to be interpreted as a physical mass [8, p.113]. In this thesis we'll work in the **modified Minimal Substraction/$\overline{\text{MS}}$ scheme**, namely what is removed is the diverging parts together with the finite constant parts obtained from the (2.6) expansion of the factor in (3.18) raised to the power of ε [8, p.114]. In practice, because it is the modified renormalization scale (3.18) that lands into (3.19), these constant finite terms never appear and therefore we only need to worry about pure ε-poles.

3.2.2 Counterterms and bare quantities

Divergences aren't removed at the Feynman diagram level but rather at the n-point function level, which is obtained by summing all of the relevant former, because ultimately that's the quantity we want to heal from infinities. As we know, n-point functions are generated from the generating functionals, which are defined thanks to the Lagrangian of the theory, therefore any cure to the n-point functions divergences must take place at the Lagrangian level. We now take advantage of a crucial property of $\mathcal{L}$, namely that it can be arbitrarily modified as long as all its symmetries are preserved. Indeed, even before any modifications, the initial Lagrangian was obtained by requiring certain symmetries to be respected and, after an arbitrary choice of fields and parameters, all its operators were determined [8, p.104]. However, one could have very well started with different fields and different parameters, obtaining a different Lagrangian with potentially different operators, which would therefore be linked to the previous one via fields and parameter redefinitions. Hence, any Lagrangian that leads to ill-defined n-point functions can be modified, provided the right redefinitions are found, with the objective to obtain one that leads to finite n-point functions. Concretely, we modify the Lagrangian as

$$\mathcal{L} \to \mathcal{L} + \mathcal{L}_{\text{CT}}, \tag{3.40}$$

where we say that $\mathcal{L}_{\text{CT}}$ contains a finite amount of **counterterms**. That is because we'll artificially insert new operators in $\mathcal{L}_{\text{CT}}$, which will produce new Feynman rules, such that these new Feynman diagrams will also diverge but, and that's the whole point, in an exactly opposed manner to the original divergences. Eventually, when summing up the old and new diagrams in the n-point functions, the overall divergences will cancel in this new, renormalized, theory, and we'll say that the original theory was renormalizable. Of course, there might not exist an $\mathcal{L}_{\text{CT}}$ composed of finite counterterms such that divergences cancel, in which case the original theory is said to be non-renormalizable [8, p.103]. However, as we'll see in Chapter 4, there's a way around. From now on, in the context of renormalization, we'll only consider 1PI n-point functions, hence only 1PI diagrams, because if we can cure the divergences for them then we'll have automatically cured them for all the other non-1PI connected diagrams and hence for all n-point functions.

Consider a theory with operators $\{O_j\}$ associated to their coupling constants $\{C_j\}$, then we partition its Lagrangian as

$$\mathcal{L} = \sum_j C_j O_j = \sum_{\text{fields}} C_f O_f + \left(\sum_{\text{mass terms}} C_i O_i + \sum_{\text{vertices}} C_i O_i \right), \tag{3.41}$$

where j runs across all kinetic terms and interaction vertices, f runs only across the fields of the theory, namely the derivative parts of the kinetic terms, and i runs through all the remaining operators. If the functional form of the counterterms is the same as the ones in the original Lagrangian, the theory is said to be exactly renormalizable [8] and we have

$$\mathcal{L} + \mathcal{L}_{\text{CT}} = \sum_j C_j O_j + \sum_j \hat{C}_j O_j = \sum_j (C_j + \hat{C}_j) O_j. \tag{3.42}$$

Since the counterterms have the same functional form, the only way they can cancel the divergences is by including divergent terms in their constants $\hat{C}_j$. One can then factor out $\hat{C}_j = \delta_j C_j$ the divergent and dimensionless parts $\{\delta_j\}$ from their respective parameters and, by defining the **renormalization constants** as $\{Z_j\} := \{1 + \delta_j\}$, get

$$\mathcal{L} + \mathcal{L}_{\text{CT}} = \sum_j (Z_j C_j) O_j. \tag{3.43}$$

This implies that the above renormalization procedure can be interpreted as a fields and coupling constants redefinition [8, p.116], as expected. Indeed, by defining the so-called **bare fields** and **bare parameters**, one can make them absorb the Lagrangian divergences

$$\begin{aligned} f_B &:= f\sqrt{Z_f}, \\ C_{i,B} &:= C_i \frac{Z_i}{\prod_f \sqrt{Z_f}^{N_{fi}}}, \end{aligned} \tag{3.44}$$

with N_{if} the number of f-fields in the i-interaction vertex, and get back the original functional form of the initial Lagrangian

$$\mathcal{L} + \mathcal{L}_{\text{CT}} = \sum_j C_{j,B} O_{j,B}, \tag{3.45}$$

where $O_{j,B}$ has the same functional form of O_j but it now contains bare fields $\{f_B\}$. Notice how in this new theory the divergences weren't actually removed altogether, but they were merely shifted from the n-point functions to this new Lagrangian. Indeed, the δ_j divergences ended up in the bare fields and parameters, which therefore became diverging, however that won't be a problem at all since they are both unobservables [8, p.116]. Finally, we conclude that

$$\sum_j Z_j C_j O_j = \sum_j C_{j,B} O_{j,B}. \tag{3.46}$$

The LHS is called the **renormalized Lagrangian**, the RHS the **bare Lagrangian**, the fields and parameters on the LHS will be called the **renormalized fields** and **renormalized parameters**, sometimes denoted as $C_j = C_{j,R}$, and because the LHS and RHS both contain counterterms they will both produce finite n-point functions. Crucially, since its functional

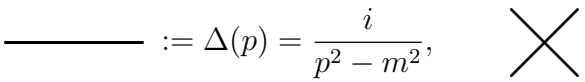
form is the same, the bare Lagrangian will produce exactly the same Feynman rules as the initial unrenormalized $\mathcal{L}$, but with bare parameters instead of renormalized ones. This is an extremely powerful result because, if we manage to find a way to convert n-point functions computed using the Feynman rules of the bare Lagrangian into n-point functions computed using the Feynman rules of the renormalized Lagrangian, then there's no need to re-compute the n-point functions with the newly added counterterms, critically the new free propagator, and all previous computations made with the unrenormalized Lagrangian can be immediately and straightforwardly recycled. To see how to convert n-point functions, simply apply (3.44) in (2.24) to obtain, after a trivial change of variables [8, p.117],

$$G^{(n)}_{\phi_{i_1,B}\dots\phi_{i_n,B}} = \sqrt{Z_{\phi_{i_1}}\cdots Z_{\phi_{i_n}}}\, G^{(n)}_{\phi_{i_1}\dots\phi_{i_n}}. \tag{3.47}$$

The same reasoning can be applied to convert 1PI n-point functions, namely by applying (3.44) to (2.26) one gets [8, p.119]

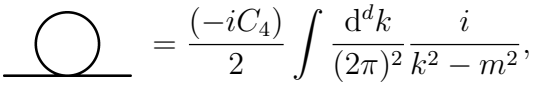
$$\Gamma^{(n)}_{\phi_{i_1,B}\dots\phi_{i_n,B}} = \sqrt{Z_{\phi_{i_1}}\cdots Z_{\phi_{i_n}}}^{(-1)}\, \Gamma^{(n)}_{\phi_{i_1,B}\dots\phi_{i_n,B}}. \tag{3.48}$$

As a practical example of the above discussion, consider a theory composed of a single field ϕ of mass m with a free propagator and a 4-point interaction vertex



(3.49)

Let's look at the 1 loop contributions to the 1PI 2-point function $\Gamma^{(2)}$ of this yet unnormalized theory. The only diagram appearing in $\Gamma^{(2)}$ is the so-called tadpole

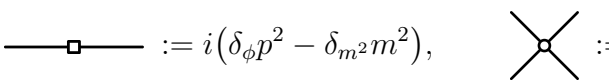


(3.50)

which is nothing but (3.20) up to multiplicative constants, therefore we know it diverges. If we now renormalize the theory by adding a counterterm in $\mathcal{L}_{\text{CT}}$ for each term in $\mathcal{L}$, we obtain a renormalization constant for the field ϕ, the mass m and the coupling constant C_4, namely Z_ϕ, Z_{m^2} and Z_{C_4}. As a result, the free propagator gets modified $\Delta(p) \to \Delta_R(p)$ as [15, p.325]

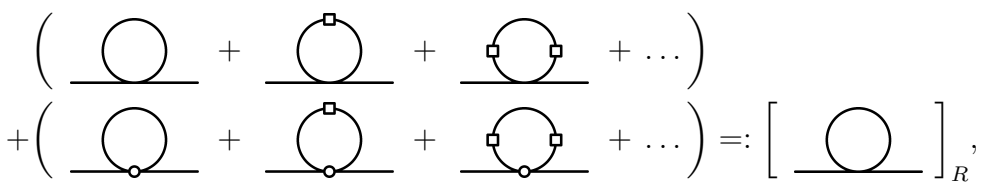
$$\begin{aligned}
\Delta_R(p) &:= \frac{i}{Z_\phi p^2 - Z_{m^2} m^2} = \frac{1}{p^2 - m^2 + (\delta_\phi p^2 - \delta_{m^2} m^2)} = \frac{i}{p^2 - m^2}\frac{1}{1 + \frac{\delta_\phi p^2 - \delta_{m^2} m^2}{p^2 - m^2}} \\
&= \frac{i}{p^2 - m^2}\left[1 + i^2\frac{\delta_\phi p^2 - \delta_{m^2} m^2}{p^2 - m^2} + i^4\left(\frac{\delta_\phi p^2 - \delta_{m^2} m^2}{p^2 - m^2}\right)^2 + \dots\right] \\
&= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} i\big(\delta_\phi p^2 - \delta_{m^2} m^2\big)\frac{i}{p^2 - m^2} \\
&\quad + \frac{i}{p^2 - m^2} i\big(\delta_\phi p^2 - \delta_{m^2} m^2\big)\frac{i}{p^2 - m^2} i\big(\delta_\phi p^2 - \delta_{m^2} m^2\big)\frac{i}{p^2 - m^2} + \dots \\
&= \quad + \quad + \quad + \dots
\end{aligned} \tag{3.51}$$

and a new interaction vertex appears



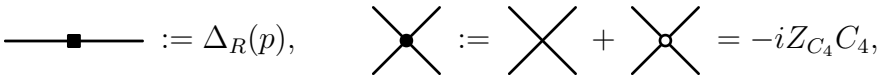
(3.52)

With those new Feynman rules, the contributions to the now renormalized 1PI 2-point function $\Gamma^{(2)}_R$ at 1 loop are



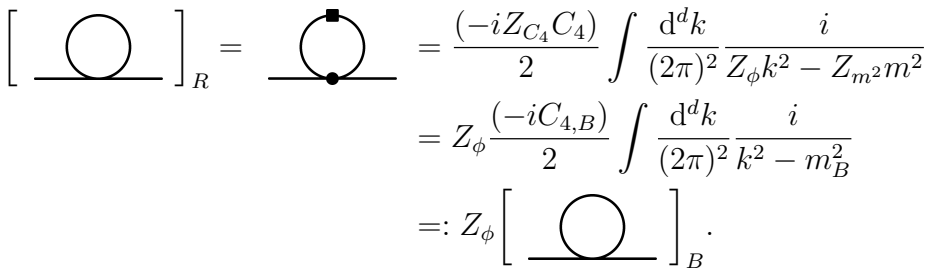
(3.53)

But if we now define



$$:= \Delta_R(p), \qquad := + = -iZ_{C_4}C_4, \tag{3.54}$$

then it becomes manifest that



$$\begin{aligned} &= \frac{(-iZ_{C_4}C_4)}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^2}\frac{i}{Z_\phi k^2 - Z_{m^2}m^2} \\ &= Z_\phi \frac{(-iC_{4,B})}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^2}\frac{i}{k^2 - m_B^2} \\ &=: Z_\phi \end{aligned} \tag{3.55}$$

In the last line, the B subscript refers to the fact that all coupling constants and all masses are to be taken as bare ones, while the R subscript means that they must all be taken as renormalized ones. With this example, we effectively showed that the renormalized tadpole can be computed by merely substituting in the unnormalized result (3.20) all the renormalized parameters with their respective bare ones, without forgetting to multiply by the appropriate amount of renormalization constants of the fields, like in (3.48).

As a final note, one must be careful while handling the modified normalization constants (3.19) because they contain masses. The bare modified normalization constant admits the following expansion

$$\bar{N}_B := \frac{1}{Q}\left(\frac{\bar{\mu}^2}{\sigma_B^2}\right)^\varepsilon = \frac{1}{Q}\left(\frac{\bar{\mu}^2}{\sigma^2}\right)^\varepsilon\left(\frac{\sigma^2}{\sigma_B^2}\right)^\varepsilon = \frac{r^\varepsilon}{Q}\left[1+\left(\frac{Z_\phi}{Z_{\sigma^2}}-1\right)\right]^\varepsilon, \tag{3.56}$$

where $r := (\bar{\mu}/\sigma)^2$, Z_{σ^2} and Z_ϕ are the renormalization constants associated, respectively, to the σ mass of the ϕ field, and where in the last equality (2.5) has to be used to further expand the square brackets bits once they land inside the logarithms arising from (2.6). Crucially, (2.6) will also produce logarithms of r in the renormalized n-point functions, which may catastrophically get multiplied with ε-poles. However, the **BPHZ theorem** states that eventually these logarithmic poles have to cancel among each other [16]. This phenomenon is called **log-cancellation** and it can be used as a check on the correctness of the whole renormalization procedure: if after that logarithmic poles still appear in n-point functions, then something went wrong. Usual logarithms of r are of course still allowed.

3.2.3 Classification of divergences and renormalizability

How can we know which theories are renormalizable and, if they are, how can we know which n-point functions should we look at to hunt for infinities? To answer these questions, we need to first introduce a classification of the divergences of diagrams.

Consider a Feynman diagram in momentum space and, after a Wick rotation (2.8) and the use of hyperspherical coordinates, observe the asymptotic behaviour $k \to \infty$ of its integrand expressed in the form $k^{\Delta-1}$. We call Δ the **superficial degree of divergence** of that diagram [8, p.100], where the -1 is there to suppress the mass dimension of the integration measure $\mathrm{d}k$. The value of Δ allows for a classification of UV divergences:

- $\Delta > 0$ is a **power-like divergence**;
- $\Delta = 0$ is a **logarithmic divergence**;
- $\Delta < 0$ is a **superficial convergence**;

hence if a diagrams has $\Delta \geq 0$ then it means that it **superficially diverges** [8, p.100]. A few calculations involving N_{fi} can show that, for a diagram [8, p.102],

$$\Delta = d - \sum_{\text{fields}} E_f \left(s_f - 1 + \frac{d}{2} \right) + \sum_{\text{vertices}} N_i \Delta_i, \tag{3.57}$$

where E_f is the number of external legs of the f-field, $s_f = 0$ if f is a boson and $s_f = 1/2$ if f is a fermion, N_i is the number of the i-vertices present in the diagram and Δ_i is the **index of divergence** of the i-vertex, which can be computed as [8, p.102]

$$\Delta_i := d_i - d + \frac{d-2}{2}(n_B)_i + \frac{d-1}{2}(n_F)_i, \tag{3.58}$$

where $(n_B)_i$ and $(n_F)_i$ are the number of bosons and fermions involved in the i-vertex and d_i is the number of derivatives found in O_i. Notably, Δ_i depends only on the functional form of O_i, therefore it is a global property of the theory [8, p.102]. Lastly, it can be shown that $\Delta_i = -[C_i]$ [8, p.102].

With that at hand, we are now able to determine which n-point functions would be divergent, namely by looking at the superficial degrees of divergence (3.57) of all of its diagrams and asking for $\Delta \geq 0$. Regarding the renormalizability condition, it can be shown that a theory is renormalizable if [8, p.103]

$$\forall i : \Delta_i \leq 0 \quad \Leftrightarrow \quad \forall i : [C_i] \geq 0 \quad \Leftrightarrow \quad \forall i : [O_i] \leq d, \tag{3.59}$$

where the first equivalence is an immediate consequence of the previous result and the second equivalence stems from the fact that $0 = [S] = [\mathcal{L}] + [\mathrm{d}^d x]$, hence $0 = [C_j] + [O_j] - d$. In particular, the last equivalence teaches us that a theory is renormalizable if all of its operators are of dimension lower or equal to $d \to 4$. Lastly, the previous argument uses mass dimensions to determine renormalizability, hence we now give a non-exhaustive list of important mass dimensions, to be later used in the rest of this thesis:

$$\begin{aligned} &[\mathrm{d}^d x] = [x]d = -d, && [\delta^d(x)] = d, && [\partial] = [m] = 1, \\ &[\mathrm{d}^d k] = [k]d = d, && [\delta^d(k)] = -d, && [f] = s_f - 1 + \frac{d}{2} = 1 + s_f - \varepsilon. \end{aligned} \tag{3.60}$$

The mass dimension of $[f]$ is obtained by computing the mass dimension of [8, p.102]

$$\Delta(x,y) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \Delta(k) e^{-ik(x-y)}, \tag{3.61}$$

where $\Delta(k) \sim k^{-2+2s_f}$, hence $2[f] = 2(s_f - 1) + d$.

3.2.4 Renormalization group equations

In the $\overline{\text{MS}}$ scheme, the renormalized 1PI 2-point function $\Gamma_R^{(2)}$, hence the full propagator $G_R^{(2)}$, will contain (finite) logarithms of r after renormalization. This means that the full propagator pole, corresponding to the physical mass, will be a function of the renormalized parameters and of the modified renormalization scale $\bar{\mu}$ [8, p.117]. However, because the physical mass must be $\bar{\mu}$-invariant, then the renormalized parameters (and fields) must be $\bar{\mu}$-dependent such to compensate the logarithms of $r = r(\bar{\mu})$ [8, p.117]. The renormalized parameters are said to be **running** $C_j = C_j(\mu)$, and because of that the renormalization constants will have an implicit $\bar{\mu}$-dependence through them [8, p.117]. The running of the renormalized parameters is obtained by solving the so-called **Renormalization Group Equations (RGEs)**

$$0 = \frac{\mathrm{d}C_{i,B}}{\mathrm{d}\ln(\mu)} = \mu\frac{\mathrm{d}C_{i,B}}{\mathrm{d}\mu} = \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\left(C'_i\mu^{[C_i]}\frac{Z_i}{\prod_f \sqrt{Z_f}^{N_{fi}}}\right) = \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\left(C'_i\mu^{[C_i]}\tilde{Z}_i\right), \quad (3.62)$$

where we defined the **extended renormalization constants** $\{\tilde{Z}_i\}$. The RGEs are equal to zero because the diverging bare parameters (and fields) must be μ-independent due to the μ-invariance of the bare Lagrangian. Contrary to that the renormalized parameters do run, and their running is quantified by either the so-called **beta functions** [13, p.140]

$$\beta_{C_i} := \frac{\mathrm{d}C'_i}{\mathrm{d}\ln(\mu)}, \quad (3.63)$$

if C_i is a coupling constant, or by the **anomalous dimensions** [13, p.140]

$$\gamma_{m_i} := \frac{\mathrm{d}\ln(m_i)}{\mathrm{d}\ln(\mu)} = \frac{1}{m_i}\frac{\mathrm{d}m_i}{\mathrm{d}\ln(\mu)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\ln\left(\tilde{Z}_{m_i}^{-1}\right), \quad (3.64)$$

if $C_i = m_i$ is a mass, which can also be viewed as a beta function

$$\beta_{m_i^2} := \frac{\mathrm{d}m_i^2}{\mathrm{d}\ln(\mu)} = 2m_i\frac{\mathrm{d}m_i}{\mathrm{d}\ln(\mu)} = 2m_i^2\gamma_{m_i}. \quad (3.65)$$

One can also associate anomalous dimensions to fields, that is [13, p.140]

$$\gamma_f := \frac{\mathrm{d}\ln\left(\sqrt{Z_f}\right)}{\mathrm{d}\ln(\mu)} = \frac{1}{2}\frac{\mathrm{d}\ln(Z_f)}{\mathrm{d}\ln(\mu)} = \sum_i \frac{\beta_{C_i}}{2}\frac{\mathrm{d}\ln(Z_f)}{\mathrm{d}C_i}. \quad (3.66)$$

In general, because in principle all renormalized parameters can appear inside the renormalization constants, the set of all RGEs (3.62) of a theory has to be solved as a system of equations, where the unknowns are the beta functions and the anomalous dimensions. Then, because in principle the latter can also contain all the renormalized parameters, a system of differential equations has yet to be solved to obtain the actual running of the parameters. That being said, in this thesis, we'll only determine the beta functions and the anomalous dimensions of the various considered theories by inverting the former system of equations.

3.2.5 Integrity check equations

Lastly, in this section we show how the obtained beta functions and anomalous dimensions can be verified against the constraint that they must satisfy the RGEs (3.62). This will provide a checking equation that has to be satisfied for all the coefficients of the beta functions and anomalous dimensions. It is worth noting that this method is the one traditionally used to obtain the $\{\beta_{C_i}\}$ and $\{\gamma_{m_i}\}$ in the first place, but since here we get them by looking at the RGEs as a system of equations, then the former reduces to a verification method.

As we said, $\{\beta_{C_i}\}$ and $\{\gamma_{m_i}\}$ contain coefficients, that is, they can be expressed as a power series of the parameters. We expect that because of the renormalization procedure by counterterms: at each loop order they will cancel divergences, therefore the $\{\delta_i\}$ will turn up to be a series in the parameters, with said divergences as their coefficients. This implies that the extended normalization constants can be expressed as

$$\tilde{Z}_i = \sum_{\lambda=0}^{\infty} \frac{\tilde{z}_\lambda^i}{\varepsilon^\lambda} = \sum_{\lambda=0}^{\infty} \frac{1}{\varepsilon^\lambda} \big(\tilde{z}_\lambda^i\big)_{\mu_1\dots\mu_n} (C_1'^{\mu_1} \cdots C_n'^{\mu_n}), \tag{3.67}$$

where we assume a theory of n dimensionless parameters, an implicit summation through $\mathbb{Z}^n$ of the $(\mu_1, \dots, \mu_n)$ indices and that, for any i-index, the coefficients obey $\tilde{z}_0^i = 1$ and $(\tilde{z}_0^i)_{\mu_1\dots\mu_n} = \delta_{\mu_1,0} \cdots \delta_{\mu_n,0}$. Analogously, for the beta functions and anomalous dimensions $\beta_i \in \{\beta_{C_i}\} \cup \{\gamma_{m_i}\}$, we propose the ansatz

$$\beta_i = \varepsilon\beta_i^\varepsilon + \bar{\beta}_i = \varepsilon\beta_i^\varepsilon + \big(\bar{\beta}_i\big)_{\mu_1\dots\mu_n} (C_1'^{\mu_1} \cdots C_n'^{\mu_n}). \tag{3.68}$$

We stress that both power series are constructed out of parameters, and not only out of coupling constants, where the dimensionfullness of the masses has to be cancelled by ratios with quantities q such that $[q] = 1$ because, we remind, the $\{\delta_i\}$ are dimensionless. That way, all the (implicit) μ-dependence of the $\{\tilde{Z}_i\}$ is captured inside the n parameters, and as a result the log-derivatives of the coefficients of both series are null. By defining $n_i := [C_i]/\varepsilon$, we can now inject the two series inside (3.62) to obtain

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)} C_i' \tilde{Z}_i \mu^{n_i\varepsilon} \\
&= \beta_i \tilde{Z}_i \mu^{n_i\varepsilon} + \frac{\partial \tilde{Z}_i}{\partial C_k'} \beta_k C_i' \mu^{n_i\varepsilon} + n_i\varepsilon C_i' \tilde{Z}_i \mu^{n_i\varepsilon} \\
&= \sum_{\lambda=0}^{\infty} \left[\frac{1}{\varepsilon^\lambda} \left(\tilde{z}_\lambda^i \bar{\beta}_i + C_i' \frac{\partial \tilde{z}_\lambda^i}{\partial C_k'} \bar{\beta}_k \right) + \frac{1}{\varepsilon^{\lambda-1}} \left(\tilde{z}_\lambda^i \beta_i^\varepsilon + C_i' \frac{\partial \tilde{z}_\lambda^i}{\partial C_k'} \beta_k^\varepsilon + n_i C_i' \tilde{z}_\lambda^i \right) \right] \\
&= \varepsilon(\beta_i^\varepsilon + n_i C_i') + \sum_{\lambda=0}^{\infty} \frac{1}{\varepsilon^\lambda} \left(\tilde{z}_{\lambda+1}^i \beta_i^\varepsilon + \tilde{z}_\lambda^i \bar{\beta}_i + C_i' \frac{\partial \tilde{z}_{\lambda+1}^i}{\partial C_k'} \beta_k^\varepsilon + C_i' \frac{\partial \tilde{z}_\lambda^i}{\partial C_k'} \bar{\beta}_k + n_i C_i' \tilde{z}_{\lambda+1}^i \right),
\end{aligned} \tag{3.69}$$

where in the last line we reshuffled the series such to order its terms by their respective power of ε. Because this series is null, all of its coefficients must be so, therefore we deduce from the linear term that $\beta_i^\varepsilon = -n_i C_i'$, which can be immediately injected inside the remaining of the series to obtain

$$0 = \tilde{z}_\lambda^i \bar{\beta}_i - n_k C_i' C_k' \frac{\partial \tilde{z}_{\lambda+1}^i}{\partial C_k'} + C_i' \frac{\partial \tilde{z}_\lambda^i}{\partial C_k'} \bar{\beta}_k, \tag{3.70}$$

valid for every $\lambda \in \mathbb{N}$. This recurrence relation can be used to verify the correctness of all the beta functions by plugging in the $\{\tilde{z}_\lambda^i\}$ coefficients, where we see that by all we truly

mean that all of them have to be considered, due to the implicit k-summation. (3.70) also shows a remarkable fact: if one sets $\lambda = 0$, then the entirety of $\bar{\beta}_i$, hence of β_i, is determined by $\tilde{z}_1^i$, namely the total $1/\varepsilon$-pole of $\tilde{Z}_i$ [8, p.121]. What's even more spectacular is that, since this is a recurrence relation, higher order total poles are determined by their predecessors, meaning that all the information is truly contained in solely the total $1/\varepsilon$-pole [13, p.146]. We stress here that all of this is valid for the total $1/\varepsilon$-pole, that is, the one obtained after adding all contributions from all loop orders.

A somewhat even more general formula than (3.70) can be obtained. Indeed, one can also take into account the $\{(\tilde{z}_\lambda^i)_{\mu_1\dots\mu_n}\}$ and $\{(\bar{\beta}_i)_{\mu_1\dots\mu_n}\}$ coefficients and expand (3.70) even further to obtain

$$\begin{aligned}
0 &= \big(\tilde{z}_\lambda^i\big)_{\mu_1\dots\mu_n}\big(\bar{\beta}_i\big)_{\nu_1\dots\nu_n}(C_1'^{\mu_1+\nu_1}\cdots C_n'^{\mu_n+\nu_n}) \\
&\quad - n_k\mu_k\big(\tilde{z}_{\lambda+1}^i\big)_{\mu_1\dots\mu_n}(C_1'^{\mu_1}\cdots C_i'^{\mu_i+1}\cdots C_n'^{\mu_n}) \\
&\quad + \mu_k\big(\tilde{z}_\lambda^i\big)_{\mu_1\dots\mu_n}\big(\bar{\beta}_k\big)_{\nu_1\dots\nu_n}(C_1'^{\mu_1+\nu_1}\cdots C_i'^{\mu_i+\nu_i+1}\cdots C_k'^{\mu_k+\nu_k-1}\cdots C_n'^{\mu_n+\nu_n}) \\
&= \big(\tilde{z}_\lambda^i\big)_{\mu_1\dots\mu_n}\big(\bar{\beta}_i\big)_{\nu_1\dots\nu_n}(C_1'^{\mu_1+\nu_1}\cdots C_n'^{\mu_n+\nu_n}) \\
&\quad - n_k(\lambda_k-\delta_{ik})\big(\tilde{z}_{\lambda+1}^i\big)_{\lambda_1\dots(\lambda_i-1)\dots\lambda_n}(C_1'^{\lambda_1}\cdots C_n'^{\lambda_n}) \\
&\quad + (\mu_k+1-\delta_{ik})\big(\tilde{z}_\lambda^i\big)_{\mu_1\dots(\mu_i-1)\dots(\mu_k+1)\dots\mu_n}\big(\bar{\beta}_k\big)_{\nu_1\dots\nu_n}(C_1'^{\mu_1+\nu_1}\cdots C_n'^{\mu_n+\nu_n}) \\
&= \Bigg\{\sum_{l_1=0}^{\lambda_1}\cdots\sum_{l_n=0}^{\lambda_n}\Big[\big(\tilde{z}_\lambda^i\big)_{l_1\dots l_n}\big(\bar{\beta}_i\big)_{(\lambda_1-l_1)\dots(\lambda_n-l_n)} \\
&\quad + (l_k+1-\delta_{ik})\big(\tilde{z}_\lambda^i\big)_{l_1\dots(l_i-1)\dots(l_k+1)\dots l_n}\big(\bar{\beta}_k\big)_{(\lambda_1-l_1)\dots(\lambda_n-l_n)}\Big] \\
&\quad - n_k(\lambda_k-\delta_{ik})\big(\tilde{z}_{\lambda+1}^i\big)_{\lambda_1\dots(\lambda_i-1)\dots\lambda_n}\Bigg\}(C_1'^{\lambda_1}\cdots C_n'^{\lambda_n}).
\end{aligned} \tag{3.71}$$

In the second equality we shifted $\mu_i \to \mu_i - 1$, $\mu_k \to \mu_k + 1$ and in its second line we renamed $\mu_i \to \lambda_i$. Notice the presence of Kronecker deltas to account for the $j = k$ case. In the third equality we used the general resummation property

$$\sum_{ij} a_{ij}b_{i+j} = \sum_\lambda\sum_{l=0}^{\lambda} a_{l,\lambda-l}b_\lambda, \tag{3.72}$$

which implies that, after adding the two λ and l sums, one must rename the original indices as $i \to l$, $j \to \lambda - l$ and $(i+j) \to \lambda$. Then, its application in the third equality leads to the renaming of $\mu_i \to l_i$, $\nu_i \to \lambda_i - l_i$ and $(\mu_i + \nu_i) \to \lambda_i$. As before, because the whole series is null, then its coefficients must be so, hence for every $(\lambda, \lambda_1, \dots \lambda_n) \in \mathbb{N} \times \mathbb{Z}^n$ it must be

$$\begin{aligned}
0 = \sum_{l_1=0}^{\lambda_1}\cdots\sum_{l_n=0}^{\lambda_n}\Big[&\big(\tilde{z}_\lambda^i\big)_{l_1\dots l_n}\big(\bar{\beta}_i\big)_{(\lambda_1-l_1)\dots(\lambda_n-l_n)} \\
&+ (l_k+1-\delta_{ik})\big(\tilde{z}_\lambda^i\big)_{l_1\dots(l_i-1)\dots(l_k+1)\dots l_n}\big(\bar{\beta}_k\big)_{(\lambda_1-l_1)\dots(\lambda_n-l_n)}\Big] \\
&- n_k(\lambda_k-\delta_{ik})\big(\tilde{z}_{\lambda+1}^i\big)_{\lambda_1\dots(\lambda_i-1)\dots\lambda_n}.
\end{aligned} \tag{3.73}$$

While this formula relies directly on the coefficients of $\bar{\beta}_i$ and $\tilde{Z}_i$, it is manifestly more cumbersome than (3.70). This is because it explicitly separates contributions by their parameter powers, hence ultimately by their loop order, while in (3.70) a harsh truncation to the desired order of the obtained result is sufficient. For this reason, only the former formula will be used in this thesis.

Chapter 4

Effective Field Theories

4.1 Motivation

In principle, every physical theory aims to describe the largest possible amount of already known phenomena. However this hope is clearly never accomplished: implicitly, every theory is experimentally valid only in the energy range set by the lowest-energy and highest-energy verified phenomenon. This is true even when a theory successfully describes all known phenomena, because what lies outside the current experimental verification might very well disagree with the theory. In this context, when a theory A has an energy range included in a larger one of a more general theory B, we say that A is an **effective theory** of B. If B is the most general known theory, B can be seen as an effective theory of a more general yet unknown theory C.

The classical example is the one where A is the Newtonian theory of gravity, applicable up to light planetary masses, B is General Relativity, applicable up to galactic masses, and C is some hypothetical beyond General Relativity theory [17, p.2]. Another historical example could be A being the Fermi interaction, postulating a 4-fermions interaction vertex, and B being the later discovered electroweak interaction, based on the exchange of $Z, W^{\pm}$ bosons. This last example is instructive: processes with energies below the mass of the lightest $W^{\pm}$ bosons are remarkably well described by the effective theory, and that is because at that scale there's just not enough energy to generate virtual $W^{\pm}$ bosons. Because they never appear, $W^{\pm}$ propagators in Feynman diagrams can be shrunk down to a degenerate point, and as a result the Fermi interaction arises as an effective vertex, as depicted in figure 4.1 [12, p.10]. However, as soon as we look at processes over that energy threshold, Fermi theory evidently falls short.

In these last 2 examples we encountered effective theories that are called **bottom-up**: this is because, when theories A were formulated, their respective theories B were unknown,

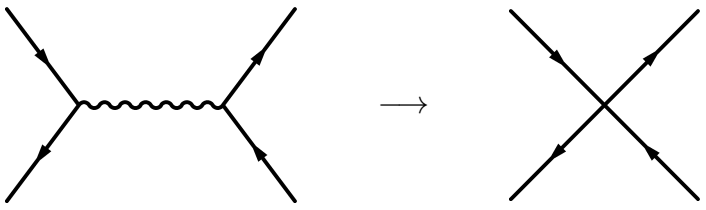


Figure 4.1: On the left, 4 fermions exchanging a $W^{\pm}$ boson through the electroweak interaction. On the right, the 4-fermions Fermi interaction. The arrow symbolizes a decrease of total energy of the external fermions.

and hence physicists were forced to work in the bottom-up A theories. A posteriori one could go in the opposite direction and, knowing theories B, voluntarily go back to their respective theories A: these effective theories are called **top-down**. There are multiple reasons why it might be interesting to work in top-down effective theories: the theory B could just be practically impossible to handle in its low-energy limit, theory A might be significantly easier to handle than theory B and finally, if multiple energy scales span theory B, it can be split into a sequence of top-down theories, each with a cutoff energy corresponding to a mass scale, allowing, if we are working with **effective field theories** (**EFTs**), not to spoil perturbation theory with gigantic logarithms of energy scale ratios [17, p.2-3].

4.2 Mathematical formulation

We now show, in a top-down scheme, how to link a particular general (or full) field theory to one of its EFTs: this will give us insights about the way forward for general full theories and bottom-up schemes.

Before anything, we have to specify the energy range over which the EFT is valid. If we call Λ the **cutoff energy** that defines the validity of the EFT, then all processes of energy E such that $E \ll \Lambda$ are faithfully described by the EFT. In field theories, it is often set $\Lambda = M$, with M the lightest mass of the set of heavy particles present in the full theory that we want to exclude from the EFT [17, p.2].

4.2.1 A particular case

Next, consider the particular case of a full theory composed of scalar fields which can be divided into 2 subsets of soft $\vec{\phi}_S$ and hard $\vec{\phi}_H$ fields. Its (unnormalized) generating functional reads [17, p.4]

$$Z[\vec{\mathbf{J}}_H, \vec{\mathbf{J}}_S] := \int \mathrm{D}\vec{\phi}_H \, \mathrm{D}\vec{\phi}_S \, \exp\left\{ i \left[S[\vec{\phi}_H, \vec{\phi}_S] + \int \mathrm{d}^d x \left(\vec{\mathbf{J}}_H \cdot \vec{\phi}_H + \vec{\mathbf{J}}_S \cdot \vec{\phi}_S \right)(x) \right] \right\}, \tag{4.1}$$

where $\vec{\mathbf{J}}_H, \vec{\mathbf{J}}_S$ are the sources associated to their respective fields. In this particular example, we wish to remove the hard fields $\vec{\phi}_H$ associated to heavy particles and only keep the soft fields $\vec{\phi}_S$, hence we define the **Wilsonian action** S_{W} to be that quantity such that [17, p.4]

$$\exp\left(i S_{\mathrm{W}}[\vec{\phi}_S] \right) := \int \mathrm{D}\vec{\phi}_H \, \exp\left(i S[\vec{\phi}_H, \vec{\phi}_S] \right). \tag{4.2}$$

Heavy fields are therefore integrated out. Now comes a crucial observation: if we consider processes for which $E \ll \Lambda$, then the diagrams appearing in the EFT will be those found in the full theory minus the n-point functions that have heavy fields as external legs or that have virtual heavy fields inside them. This means that no functional derivatives of heavy fields over the full generating functional will ever be taken, therefore one can just set $\vec{\mathbf{J}}_H = \vec{\mathbf{0}}$ and, using (4.2), obtain [12, p.6]

$$Z[\vec{\mathbf{0}}, \vec{\mathbf{J}}_S] = \int \mathrm{D}\vec{\phi}_S \, \exp\left[i \left(S_{\mathrm{W}}[\vec{\phi}_S] + \int \mathrm{d}^d x \, \vec{\mathbf{J}}_S(x) \cdot \vec{\phi}_S(x) \right) \right]. \tag{4.3}$$

4.2.2 General case

Equation (4.3) would allow us to exactly compute the EFT, however (4.2) is in practice impossible to calculate [17, p.4]. Therefore, the way forward is clearly some sort of approximation of (4.3) that doesn't require (4.2) and that hence also works in cases where one can't

directly partition the fields of the full theory into soft and hard modes. To do so, define the EFT (unnormalized) generating functional [17, p.4]

$$Z_{\text{EFT}}[\vec{\mathbf{J}}] := \int \mathrm{D}\vec{\phi}\, \exp\left[i\left(S_{\text{EFT}}[\vec{\phi}] + \int \mathrm{d}^d x\, \vec{\mathbf{J}}(x)\cdot\vec{\phi}(x)\right)\right], \tag{4.4}$$

where $\vec{\phi} \neq \vec{\phi}_S$ are totally unrelated and S_{EFT} is the spacetime integral of the Lagrangian

$$\mathcal{L}_{\text{EFT}} := \mathcal{L}_{[O_j]\leq d} + \sum_{[O_j]>d} \frac{C_j}{\Lambda^{[O_j]-d}} O_j + \mathcal{L}_{\text{CT}}, \tag{4.5}$$

where the C_j are called **Wilson coefficients** and they are divided by an appropriate power of Λ such that the former are dimensionless $[C_j] = 0$ at $d = 4$ [17, p.5]. In the above, the first term indicates the portion of $\mathcal{L}_{\text{EFT}}$ composed of operators O_j of dimension $[O_j] \leq d$, the second term indicates a tower of operators O_j of increasing dimension $[O_j] > d$, called an **Operator Product Expansion (OPE)** and justified in [18], and the third term indicates the counterterms needed to renormalize the theory in the EFT sense. Indeed, the presence of $[O_j] > d$ operators renders the theory non-renormalizable in the usual sense, therefore we must define a new notion of renormalizability such that operators of dimensions $[O_j] > d$ are allowed and n-point functions stay finite [17, p.3]. This is done by defining the **expansion parameter** [17, p.6]

$$\delta := \frac{q}{\Lambda}, \tag{4.6}$$

where q is a quantity, generally a mass or a momentum, such that $[q] = [\Lambda] = 1$. Recalling $E \ll \Lambda$, it is clear that it must be $\delta \ll 1$, therefore δ can indeed be used as an expansion parameter. We'll say that a theory is renormalizable in the EFT sense if it is renormalizable in the usual sense up to order $[O] - d$ in δ, where $[O] := \max [O_j]$ indicates the maximum operator dimension of the, we conclude, truncated OPE. Every quantity of order $\mathcal{O}\big(\delta^{[O]-d+1}\big)$ will have to be discarded from any calculations: this ensures that the number of counterterms needed in $\mathcal{L}_{\text{CT}}$ stays finite and that, with this prescription, the usual renormalization procedure can be applied, as long as we are willing to accept an error of $\mathcal{O}\big(\delta^{[O]-d+1}\big)$ in the n-point functions [12, p.19]. From now on, we'll omit to write the additional $\mathcal{O}\big(\delta^{[O]-d+1}\big)$ term in the OPE, indicating that it is truncated, because it'll be assumed to be implicit.

With our newly defined $\mathcal{L}_{\text{EFT}}$, containing operators of dimension at most $[O_j] \leq [O]$, we now have all the elements to formulate the **matching condition** [17, p.4]

$$Z_{\text{EFT}}[\vec{\mathbf{J}}] + \mathcal{O}\big(\delta^{[O]-d+1}\big) = Z[\vec{\mathbf{0}}, \vec{\mathbf{J}}], \tag{4.7}$$

ensuring that the EFT constructed out of the OPE is indeed the EFT of the full theory, up to errors of order $\mathcal{O}\big(\delta^{[O]-d+1}\big)$. As a final remark, terms in $\mathcal{L}_{[O_j]\leq d}$ can, relaxing the condition on the dimension, formally also be considered part of the OPE.

4.2.3 Matching procedure

We now ask ourselves what can be said about Wilson coefficients. In bottom-up schemes, they are either left undefined or empirically measured. In top-down schemes, however, we can directly link them with the original parameters of the full theory via a procedure called **matching** [17, p.15], which we will not outline in this thesis but that can be derived from (4.7), such that every Wilson coefficient can be expressed in function of the full theory

parameters. Because Wilson coefficients are parameters of a theory, they must also run $C_j = C_j(\mu)$ according to some RGEs, therefore one must decide at which renormalization scale μ the matching should be done. Since in the full theory Λ scales appear as natural mass scales in its n-point functions, the matching is done at the boundary of validity of the EFT $\mu_1 = \Lambda$ and then, thanks to the RGEs, the matched Wilson coefficients are run down $\mu_1 \to \mu_2 < \mu_1$ to a lower, more suitable, renormalization scale $\mu_2 = E$ of the order of the processes described by the EFT, so that they can be used there [17, p.12]. A particularity of said RGEs is that a Wilson coefficient at scale μ_2 is not only a function of itself at scale μ_1 but also of other Wilson coefficients at that scale, hence in the descent $\mu_1 \to \mu_2$ we say that **mixing** happens among Wilson coefficients [17, p.13].

4.3 Operator Product Expansion

In order to explicitly construct an EFT, all operators O_j in (4.5) must be explicitly known up to the desired dimension. As they depend on the full theory, there's no general formula to derive them: one is therefore forced to consider all of the possible operators and assume all Wilson coefficients, until proven otherwise, to be $C_j \neq 0$ [17, p.5]. What we can say, however, is what kind of constraints the O_j must satisfy to be physically and mathematically coherent: this reduces, sometimes significantly, the number of operators O_j that have to be considered.

4.3.1 Constructing the OPE

We identify the following conditions that O_j must satisfy [17, p.2-3]:

- **Lorentz invariance**: they must stay the same under Lorentz transformations;
- **Symmetry invariance**: they must respect the inherited symmetries of the full theory;
- **Redundancy**: they must not be redundant with any other operators in the OPE.

We say an operator is **redundant** if there exists an operation that transforms it into an equivalent linear combination of other different operators. With this definition, we'll call an **(operator) basis** the minimal set of non-redundant operators of an EFT, and we'll abusively consider the OPE to be constructed out of the elements of this basis. How can operators be redundant? The trivial example is through integrations by parts, since total derivatives do not contribute to S_{EFT} [17, p.8]. The basis obtained after removing all operators redundant this way is called the **Green's basis** and it is denoted by $G^{(d)}_{\text{EFT}}$, with $d = [O]$ the dimension of the basis. The second possibility is through field redefinitions [17, p.8], which will be discussed below. The basis obtained after removing from the Green's basis all operators redundant this way is called the **physical basis**, and it is denoted by $P^{(d)}_{\text{EFT}}$ with again $d = [O]$. Clearly the latter is a subset of the former $P^{(d)}_{\text{EFT}} \subset G^{(d)}_{\text{EFT}}$.

4.3.2 Power counting formula

Once the OPE has been established, we need to implement the expansion parameter prescription when generating Feynman diagrams in a diagrammatic way, in order to determine which ones should be kept and which ones should be discarded. Consider a generic diagram of amplitude A containing only one insertion of an O operator. Because of (4.5), it is clear that this insertion will contribute a power of $A \sim \delta^{[O]-d}$, where the appropriate q in δ will be automatically provided to get a dimensionless amplitude $[A] = 0$ at $d = 4$. Multiple

insertions of interaction operators O_i, where i runs across all the vertices in the diagram, give a contribution of [17, p.6]

$$A \sim \delta^{\sum_i ([O_i]-d)}. \tag{4.8}$$

This is called the **power counting formula**, and it can be proved that in DR it remains valid even above tree level, where one could think that integrations over loop momenta would spoil the $\delta \ll 1$ condition [17, p.6].

Applying the power counting formula allows us to see that, for example, a graph with two insertions of 6-dimensional operators would be of order δ^4, the same as one single insertion of an 8-dimensional operator. If for example we are interested in an EFT of $\mathcal{O}(\delta^3)$, then all operators of dimension higher or equal than 7 should be excluded from the OPE, and in Feynman diagrams only one single insertion of 6-dimensional operators is authorised at any loop order. In this thesis we'll be interested in exactly that, therefore from now on an implicit $\mathcal{O}(\delta^3) = \mathcal{O}(\Lambda^{-3})$ cutoff prescription is implemented and only operators of dimensions 5 and 6 are considered in OPEs.

4.3.3 Shifted free propagator expansion

If the above implicit $\mathcal{O}(\Lambda^{-3})$ prescription has to be realized, then an interesting consequence for Feynman diagrams can be derived. As we said, the errors we are willing to accept in an EFT are controlled by the expansion parameter δ. This means that, should a combination p of external momenta appear in the numerator of δ, then, because of $\mathcal{O}(\Lambda^{-3}) = \mathcal{O}(\delta^3)$, terms of order $\mathcal{O}\big((p^2)^{3/2}\big)$ in the amplitudes should be discarded or, equivalently, terms of order $\mathcal{O}(p^3)$ should be discarded in the integrands. With that at hand, we can simplify the computation of any Feynman diagram containing shifted free propagators. Indeed, if k is a loop momentum shifted by a combination p of external momenta, then [12, p.22]

$$\begin{aligned}\frac{1}{(k+p)^2-\sigma^2} &= \frac{1}{k^2-\sigma^2}\frac{k^2-\sigma^2}{(k+p)^2-\sigma^2} = \frac{1}{k^2-\sigma^2}\frac{(k^2+2k\cdot p+p^2-\sigma^2)-p^2-2k\cdot p}{(k+p)^2-\sigma^2} \\ &= \frac{1}{k^2-\sigma^2}\left(1-(p^2+2k\cdot p)\frac{1}{(k+p)^2-\sigma^2}\right)\end{aligned} \tag{4.9}$$

which, applied iteratively, leads to [12, p.22]

$$\frac{1}{(k+p)^2-\sigma^2} = \frac{1}{k^2-\sigma^2} - \frac{p^2+2k\cdot p}{(k^2-\sigma^2)^2} + \frac{(2k\cdot p)^2}{(k^2-\sigma^2)^3} + \mathcal{O}\big(p^3\big). \tag{4.10}$$

We see that, by virtue of the $\mathcal{O}(\Lambda^{-3})$ prescription, the shifted free propagator decomposes into a sum of unshifted free propagators, which are easier to handle. As we'll see in the upcoming Chapter 5, this expansion will be systematically applied so that no shifted free propagators will ever appear.

4.4 Field redefinitions

In the previous section we mentioned that operators can be redundant through field redefinitions. Before explaining the meaning of the former, we discuss the latter.

A **field redefinition** is a $\vec{\phi} \to \vec{\phi}'$ transformation of the form

$$\vec{\phi} = \vec{\mathbf{F}}[\vec{\phi}'] := \vec{\phi}' + \frac{\vec{\mathbf{f}}[\vec{\phi}']}{\Lambda^n}, \tag{4.11}$$

where $\vec{\mathbf{f}}$ respects the power counting $[\vec{\mathbf{f}}] = (n+1)$ and Lorentz invariance. We remind the reader that the notation $\vec{\mathbf{f}}[\vec{\phi}]$ implies $\vec{\mathbf{f}}[\vec{\phi}] = \vec{\mathbf{f}}[\vec{\phi}, \partial\vec{\phi}, \dots, \partial^n\vec{\phi}]$, where the order of the derivatives stops at n so that $\vec{\mathbf{f}}$ respects the power counting. Because in this thesis we work at order $\mathcal{O}(\Lambda^{-3})$, it'll be $n = 2$. The reason why (4.11) is of that form is because this way we ensure the kinetic parts of the fields stay in their canonical form [19]. Next, we observe the effect of such a transformation on the (unnormalized) generating functional [2, p.51]:

$$\begin{aligned} Z[\vec{\mathbf{J}}] &= \int \mathrm{D}\vec{\phi}\, \exp\left[i \int \mathrm{d}^d x \left(\mathcal{L}[\vec{\phi}(x)] + \vec{\mathbf{J}}(x) \cdot \vec{\phi}(x) \right) \right] \\ &= \int \mathrm{D}\vec{\phi}' \left| \det\left(\frac{\delta \vec{\mathbf{F}}[\vec{\phi}']}{\delta \vec{\phi}'} \right) \right| \exp\left[i \int \mathrm{d}^d x \left(\mathcal{L}'[\vec{\phi}'(x)] + \vec{\mathbf{J}}(x) \cdot \vec{\mathbf{F}}[\vec{\phi}'(x)] \right) \right] \\ &= \int \mathrm{D}\vec{\phi}\, \exp\left[i \int \mathrm{d}^d x \left(\mathcal{L}'[\vec{\phi}(x)] + \vec{\mathbf{J}}(x) \cdot \vec{\mathbf{F}}[\vec{\phi}(x)] \right) \right]. \end{aligned} \tag{4.12}$$

In the second line of the above we defined the **modified Lagrangian** [2, p.51]

$$\mathcal{L}'[\vec{\phi}'] := (\mathcal{L} \circ F)[\vec{\phi}'] = \mathcal{L}[F[\vec{\phi}']] = \mathcal{L}[\vec{\phi}], \tag{4.13}$$

while in the third line we computed the determinant to be unity and renamed the dummy integration fields $\vec{\phi}' \to \vec{\phi}$. Indeed, to determine the value of the determinant, first expand [17, p.10]

$$\frac{\delta \vec{\mathbf{F}}[\vec{\phi}'(x)]}{\delta \vec{\phi}'(y)} = \delta(x-y)\mathbb{I} + \frac{1}{\Lambda^n}\delta(x-y)\mathbb{M}[\vec{\phi}'(x)], \tag{4.14}$$

where $\mathbb{I}$ is the $N \times N$ unit matrix (remember that there are N fields in $\vec{\phi}'$) and in the second term of the RHS we factored out the delta functions, producing the functional $N \times N$ matrix $\mathbb{M}$: this can be done because of the $z\delta'(z) = \delta(z)$ property of Dirac deltas, where $z \in \mathbb{R}$. From there, using the Faddeev-Popov idea [17, p.10]

$$\begin{aligned} \det\left(\frac{\delta \vec{\mathbf{F}}[\vec{\phi}'(x)]}{\delta \vec{\phi}'(y)} \right) &= \int \mathrm{D}\vec{\mathbf{a}} \int \mathrm{D}\vec{\mathbf{b}}\, \exp\left\{ i \int \mathrm{d}^d x \int \mathrm{d}^d y \left[\vec{\mathbf{a}}^\top(x) \left(\frac{\delta \vec{\mathbf{F}}[\vec{\phi}'(x)]}{\delta \vec{\phi}'(y)} \right) \vec{\mathbf{b}}(y) \right] \right\} \\ &= \int \mathrm{D}\vec{\mathbf{a}} \int \mathrm{D}\vec{\mathbf{b}}\, \exp\left\{ i \int \mathrm{d}^d x \left[\vec{\mathbf{a}}^\top(x) \left(\mathbb{I} + \frac{1}{\Lambda^n}\mathbb{M}[\vec{\phi}'(x)] \right) \vec{\mathbf{b}}(x) \right] \right\} = 1, \end{aligned} \tag{4.15}$$

where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are Grassmann ghost fields, we see that the determinant can be viewed as a generating functional of a theory composed of ghosts, which are associated to source terms contained in $\mathbb{M}[\vec{\phi}'(x)]/\Lambda^n$. However, we immediately notice that all ghost propagators are constants, meaning that all the generated diagrams lead to scaleless integrals, which in DR are null [17, p.10]. Finally, by looking at the Grassmann generalization of (2.27) and by setting all n-point Green's functions to zero, we obtain the desired result. Now, go back to the original generating functional and observe that the same $Z[\vec{\mathbf{J}}]$, depending on which functional derivatives are taken, can produce the equivalent n-point Green's functions

$$\langle \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n) \rangle \Big|_{\text{From } \mathcal{L}} = \Big\langle F_{i_1}[\vec{\phi}(x_1)] \dots F_{i_n}[\vec{\phi}(x_n)] \Big\rangle \Big|_{\text{From } \mathcal{L}'}, \tag{4.16}$$

where the $\vec{\mathbf{F}}[\vec{\phi}]$ are called **interpolating fields** [17, p.10]. This has to be compared with the n-point Green's functions generated by the modified (unnormalized) generating functional [17, p.10]

$$Z'[\vec{\mathbf{J}}] = \int \mathrm{D}\vec{\phi}\, \exp\left[i \int \mathrm{d}^d x \left(\mathcal{L}'[\vec{\phi}(x)] + \vec{\mathbf{J}}(x) \cdot \vec{\phi}(x) \right) \right], \tag{4.17}$$

namely [17, p.10]

$$\langle \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n) \rangle \Big|_{\text{From } \mathcal{L}'} \neq \left\langle F_{i_1}[\vec{\phi}(x_1)] \dots F_{i_n}[\vec{\phi}(x_n)] \right\rangle \Big|_{\text{From } \mathcal{L}'}. \tag{4.18}$$

Crucially, the RHS is different from the LHS because of the interpolating fields, even if both n-point Green's functions came from the same Lagrangian $\mathcal{L}'$ obtained after a field redefinition (4.11). But then how can it be that, with different n-point Green's functions, field redefinitions remove redundancies, which therefore shouldn't affect physical results? The answer lies in the **S-matrix equivalence theorem** which, for field redefinitions precisely of the form (4.11), states that S-matrix elements are left invariant [19]. Said elements can be obtained through the **LSZ reduction formula** [2, p.50]

$$\langle p_1, \dots, p_{n_{\text{out}}} | p_{n_{\text{out}}+1}, \dots, p_n \rangle = \lim_{\substack{p_i^2 \to m_i^2 \\ p_i^0 > 0}} \prod_{i=1}^{n} \left(\frac{p_i^2 - m_i^2}{i\sqrt{R_i}} \right) G^{(n)}(p_1, \dots, p_{n_{\text{out}}}, -p_{n_{\text{out}}+1}, \dots, -p_n), \tag{4.19}$$

where $n = n_{\text{in}} + n_{\text{out}}$, on the LHS all 4-vectors are on-shell and on the RHS the $n_{\text{in}} = n - n_{\text{out}}$ incoming 4-vectors have their sign flipped because in this thesis by convention all external momenta are inwards. The LSZ reduction formula links the unphysical n-point functions, hence the n-point Green's functions, to the physically measurable S-matrix elements. Therefore, the S-matrix equivalence theorem can be rephrased in this context as follows: for field redefinitions (4.11), the differences on the n-point functions are compensated by the differences in the wavefunction factors R_i in the LSZ reduction formula, such that S-matrix elements stay invariant [17, p.11]. The takeaway message is that, even if field redefinitions modify the Lagrangian $\mathcal{L} \to \mathcal{L}'$, the physical S-matrix elements won't be affected by them. We can take advantage of this fact by finding field redefinitions that transform the Lagrangian to a simpler one, that is, with fewer operators in the OPE, therefore effectively reducing some of them into, we conclude, equivalent ones. This is what we mean by redundant operators through field redefinitions. A particularly useful relation for that is

$$\partial(\phi_i \partial\phi_j) \equiv 0 \quad \Leftrightarrow \quad \phi_i \Box \phi_j \equiv -(\partial\phi_i)(\partial\phi_j), \tag{4.20}$$

which is helpful to determine the kinetic term contributions from the transformations. Moreover, because a (useful) field redefinition removes at least one redundant operator (or more, if we are lucky), then it is clear that the cardinality of the physical basis is bounded

$$\left| P^{(d)}_{\text{EFT}} \right| \leq \left| G^{(d)}_{\text{EFT}} \right| - \left| \Omega^{(d)}_{\text{EFT}} \right|, \tag{4.21}$$

where $\Omega^{(d)}_{\text{EFT}}$ is the set of all possible useful field redefinitions.

It is worth mentioning that the order of field redefinitions is relevant: if a transformation $f_{A\to B}$ reduces an operator A to another one B and a transformation $f_{B\to C}$ reduces that same operator B to yet another one C, then it is clear that a successive application of $f_{A\to B}$ and then $f_{B\to C}$ will leave us with only operator C, as opposed to a successive application of $f_{B\to C}$ and then $f_{A\to B}$ which will leave us with operators B and C. Lastly, reference [2, p.52]

shows that the use of EOMs, which can be derived using (2.20), is ultimately a particular class of field redefinitions. Although they are practical and they allow us to quickly figure out the needed field redefinitions, they should be avoided because there are cases in which they are mathematically inconsistent [2, p.56]. If however one really wants to use them, then one must be aware that the EOMs should be derived with the Lagrangian containing renormalized fields, rather than bare ones.

4.5 Beta functions

We've just seen that, thanks to the S-matrix equivalence theorem, the physically measurable S-matrix elements are left invariant by field redefinitions of the form (4.11). It is therefore legitimate to ask ourselves which other physical quantities are also left invariant by said transformations. The running of parameters is an experimentally verified phenomenon, therefore the beta functions of the latter are also physical quantities that can be measured.

However, traditional beta functions cannot possibly be left invariant by field redefinitions, as $P^{(d)}_{\text{EFT}} \subset G^{(d)}_{\text{EFT}}$, therefore the number of parameters reduces along with the number of beta functions. What however turns out to be left invariant by field redefinitions are the so-called **effective beta functions**, thanks to a result known as the **effective beta function theorem** [20]. Indeed, reference [20] shows that, as a corollary of the S-matrix equivalence theorem, the renormalization group equations of S-matrix elements can be reshaped such to contain new quantities, the effective beta functions, which describe the running of the parameters associated to the physical basis $P^{(d)}_{\text{EFT}}$. It is then proved that, despite still containing the parameters of $G^{(d)}_{\text{EFT}} \setminus P^{(d)}_{\text{EFT}}$ and their associated beta functions, the effective beta functions are only functions of the parameters associated to $P^{(d)}_{\text{EFT}}$, meaning that a cancellation among $G^{(d)}_{\text{EFT}} \setminus P^{(d)}_{\text{EFT}}$ terms must happen. Eventually it is shown that, thanks to this cancellation, obtaining the effective beta functions from the parameters and the traditional beta functions of $G^{(d)}_{\text{EFT}}$ is equivalent to directly obtaining the effective beta functions from the parameters of $P^{(d)}_{\text{EFT}}$, with the crucial caveat that the renormalization of the physical theory $P^{(d)}_{\text{EFT}}$ has to be done differently than the EFT theory $G^{(d)}_{\text{EFT}}$. Indeed, because $P^{(d)}_{\text{EFT}} \subset G^{(d)}_{\text{EFT}}$, the physical theory will lack operators that were necessary for it to be renormalizable in the EFT sense. However this won't be an issue, because we know that the missing $G^{(d)}_{\text{EFT}} \setminus P^{(d)}_{\text{EFT}}$ operators are precisely those redundant via field redefinitions, therefore the physical theory can still be renormalized as long as the necessary $G^{(d)}_{\text{EFT}} \setminus P^{(d)}_{\text{EFT}}$ counterterms are eventually removed by field redefinitions. This result simplifies enormously the diagrammatic calculations because it allows us to compute n-point functions by merely using the significantly less amount of $P^{(d)}_{\text{EFT}}$ Feynman rules. The renormalization of these n-point functions will involve proper $P^{(d)}_{\text{EFT}}$ counterterms, hence proper renormalization constants Z_j, and it will require $(G^{(d)}_{\text{EFT}} \setminus P^{(d)}_{\text{EFT}}) \not\subset P^{(d)}_{\text{EFT}}$ counterterms, outside of the Feynman rules, which at the end of the day will be absorbed inside the renormalization constants via field redefinitions [20]. To distinguish these latter **improper counterterms** from the genuine former ones, they will be denoted by either $\bar{\delta}_j$, $\hat{\delta}_j$ or $\tilde{\delta}_j$, and they obviously won't be assigned to any Z_j.

We stress once again that working with the physical theory is only possible because we are interested in the effective beta functions and, while obtaining them, the contributions coming from the EFT theory cancel among themselves: in other words, the EFT theory contains more information than the physical one, however this information is useless, therefore one doesn't need to compute it through the full EFT theory set of Feynman rules. From now on, we'll abusively refer to the effective beta functions as simply the beta functions.

Chapter 5

Symbolic computation of diagrams

5.1 Mathematical framework

5.1.1 Motivation

We are interested in automating the evaluation of Feynman diagrams through computer-algebra programs. To do so, we must devise an algorithm able to systematically evaluate integrals of the form [7, p.33]

$$J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots] := \int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D k_m}{i\pi^{D/2}} \left(\frac{[\dots k^\mu \dots]}{A_1^{\nu_1} \cdots A_n^{\nu_n}} \right), \tag{5.1}$$

where the notation $[\dots k^\mu \dots]$ means a product made out of any combination of the m loop momenta $k_1, \dots, k_m$ and $\nu_1, \dots, \nu_n \in \mathbb{N}$ are the powers over which the n internal propagators

$$A_i := \left(\sum_j \xi_{ij} k_j + \sum_l \chi_{il} p_l \right)^2 - \sigma_i^2 \tag{5.2}$$

are raised, with $i \in \{1, \dots, n\}$, where σ_i is the mass of the particle being propagated, $\xi_{ij}, \chi_{il} \in \mathbb{N}$ are arbitrary coefficients with indices $j \in \{1, \dots, m\}$, $l \in \{1, \dots, r\}$, and $p_1, \dots, p_r$ are the r external momenta. Since in this thesis we restrict ourselves to 2 loop diagrams, from now on it'll be either $m = 1$ or $m = 2$. Moreover, we'll define and call $J^D_{\nu_1\dots\nu_n} := J^D_{\nu_1\dots\nu_n}[1]$ the **scalar integral**.

5.1.2 The scalar integral

We'd like to obtain a representation of the scalar integral general to all loop orders: this will ensure a systematic treatment of both $m = 1, 2$ cases.

Consider first the $m = 2$ case. We put ourselves into the Schwinger representation by determining the coefficients $a, b, c, f \in \mathbb{R}$ and the 4-vectors d^μ, e^μ appearing in [7, p.37]

$$\sum_i x_i A_i = a k_1^2 + b k_2^2 + 2c(k_1 \cdot k_2) + 2(d \cdot k_1) + 2(e \cdot k_2) + f. \tag{5.3}$$

Notice that, by construction, all coefficients and all components of the 4-vectors are linear combinations of $x_1, \dots, x_n$. Then, diagonalize the above sum by transforming the loop momenta with the change of variables [7, p.37]

$$\begin{cases} k_1^\mu = K_1^\mu - \dfrac{c}{a} K_2^\mu + X^\mu \\ k_2^\mu = K_2^\mu + Y^\mu \end{cases} \tag{5.4}$$

where we defined [7, p.38]

$$X^\mu := \frac{ce^\mu - bd^\mu}{P}, \tag{5.5}$$

$$Y^\mu := \frac{cd^\mu - ae^\mu}{P}, \tag{5.6}$$

$$P := ab - c^2, \tag{5.7}$$

$$Q := -ae^2 - bd^2 + 2c(e\cdot d) + fP. \tag{5.8}$$

This allows us to rewrite the Schwinger sum (5.3) as [7, p.38]

$$\sum_i x_i A_i = aK_1^2 + \frac{P}{a}K_2^2 + \frac{Q}{P}, \tag{5.9}$$

which is indeed diagonalized in the loop momenta and is therefore in a suitable form to finally carry on the two momenta integrations

$$\begin{aligned}
J^D_{\nu_1\dots\nu_n} &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \int \frac{\mathrm{d}^D K_1}{i\pi^{D/2}} \frac{\mathrm{d}^D K_2}{i\pi^{D/2}} \exp\left(aK_1^2 + \frac{P}{a}K_2^2 + \frac{Q}{P}\right) \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \frac{\Omega_D^2 e^{Q/P}}{\pi^D} \int_0^\infty \mathrm{d}K_1\,\mathrm{d}K_2\, K_1^{D-1} K_2^{D-1} \exp\left(-aK_1^2 - \frac{P}{a}K_2^2\right) \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \frac{4e^{Q/P}}{\Gamma(D/2)^2} \left(\int_0^\infty \mathrm{d}K_1\, K_1^{D-1} e^{-aK_1^2}\right)\left(\int_0^\infty \mathrm{d}K_2\, K_2^{D-1} e^{-PK_2^2/a}\right) \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \frac{e^{Q/P}}{\Gamma(D/2)^2} \left(\int_0^\infty \mathrm{d}K_1^2\, (K_1^2)^{\frac{D}{2}-1} e^{-aK_1^2}\right)\left(\int_0^\infty \mathrm{d}K_2^2\, (K_2^2)^{\frac{D}{2}-1} e^{-PK_2^2/a}\right) \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \frac{e^{Q/P}}{\Gamma(D/2)^2} \frac{\Gamma(D/2)}{a^{D/2}} \frac{\Gamma(D/2)}{(P/a)^{D/2}} = \int_{\nu_1\dots\nu_n} \mathrm{D}x\, R^D,
\end{aligned} \tag{5.10}$$

where in the second line we Wick rotated the integrals and went to the usual hyperspherical coordinates, in the fourth line we massaged the integrals into the definition of the Gamma function and in the last line we defined [7, p.38]

$$R^D := \frac{e^{Q/P}}{P^{D/2}}. \tag{5.11}$$

We now show that the same procedure can be carried out for $m = 1$ and that its final result will be analogous. In Schwinger representation, we again determine the (different than in $m = 2$) coefficients $a, f \in \mathbb{R}$ and 4-vector d^μ appearing in

$$\sum_i x_i A_i = ak^2 + 2(d\cdot k) + f. \tag{5.12}$$

As before, by construction, they are all linear in the $x_1, \dots, x_n$ parameters. Now, consider the change of variable

$$k^\mu = K^\mu + X^\mu, \tag{5.13}$$

where we defined

$$X^\mu := -\frac{d^\mu}{P}, \tag{5.14}$$

$$P := a, \tag{5.15}$$

$$Q := -d^2 + fP. \tag{5.16}$$

This allows us to write the Schwinger sum (5.12) as

$$\sum_i x_i A_i = aK^2 + \frac{Q}{P}, \tag{5.17}$$

which is indeed diagonalized in the loop momentum, and therefore the loop momentum integration can be analogously executed as

$$\begin{aligned} J^D_{\nu_1\dots\nu_n} &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \int \frac{\mathrm{d}^D K}{i\pi^{D/2}} \exp\left(aK^2 + \frac{Q}{P}\right) \\ &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \, \frac{\Omega_D e^{Q/P}}{\pi^{D/2}} \int_0^\infty \mathrm{d}K \, K^{D-1} e^{-aK^2} \\ &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \, \frac{e^{Q/P}}{\Gamma(D/2)} \int_0^\infty \mathrm{d}K^2 \, (K^2)^{\frac{D}{2}-1} e^{-aK^2} \\ &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \, \frac{e^{Q/P}}{\Gamma(D/2)} \frac{\Gamma(D/2)}{a^{D/2}} = \int_{\nu_1\dots\nu_n} \mathrm{D}x \, R^D, \end{aligned} \tag{5.18}$$

giving us again the same representation of the scalar integral as in the $m = 2$ case.

5.1.3 Tensor reduction

We now want to link general integrals $J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots]$ to the scalar integral $J^D_{\nu_1\dots\nu_n}$. To do so one must somehow get rid of the $[\dots k^\mu \dots]$ information, that is, transfer the tensor indices of the loop momenta to some fixed tensor: this is called **tensor reduction**.

The way forward is given by the following identities valid for any $a \in \mathbb{R}$ [7, p.38]:

$$\begin{aligned} \int \frac{\mathrm{d}^D K}{i\pi^{D/2}} K^\mu e^{aK^2} &= 0, \\ \int \frac{\mathrm{d}^D K}{i\pi^{D/2}} K^\mu K^\nu e^{aK^2} &= -\frac{1}{2a^{D/2+1}} g^{\mu\nu}, \\ \int \frac{\mathrm{d}^D K}{i\pi^{D/2}} K^\mu K^\nu K^\rho e^{aK^2} &= 0, \\ \int \frac{\mathrm{d}^D K}{i\pi^{D/2}} K^\mu K^\nu K^\rho K^\sigma e^{aK^2} &= \frac{1}{4a^{D/2+2}} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}), \\ &\dots \end{aligned} \tag{5.19}$$

We observe that, for an odd number of momenta K, the integrals trivially vanish due to their odd parity, while for an even number of K's the integrals reduce to a constant times a specific tensor which has absorbed all the momenta indices. The latter cases can be derived by contracting a tensor ansatz with itself: for example, for the second identity, propose

$$\int \frac{\mathrm{d}^D K}{i\pi^{D/2}} K^\mu K^\nu e^{aK^2} = A T^{\mu\nu} \tag{5.20}$$

with $A \in \mathbb{R}$, $T^{\mu\nu} = g^{\mu\nu}$ and contract both sides with $T_{\mu\nu}$ to get

$$A = \frac{1}{D} \int \frac{\mathrm{d}^D K}{i\pi^{D/2}} K^2 e^{aK^2} = -\frac{1}{2a^{D/2+1}}, \tag{5.21}$$

where the integral is performed similarly as in the previous section.

Once identities (5.19) have been established, they can be applied to (5.1), if it is first brought into the correct form where the loop momenta are those given by the change of variables (5.13) if $m = 1$, or (5.4) if $m = 2$, that is

$$J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots] = \int_{\nu_1\dots\nu_n} \mathrm{D}x \int \frac{\mathrm{d}^D K_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D K_m}{i\pi^{D/2}} [\dots K^\mu \dots] \exp\left(\sum_i x_i A_i\right). \tag{5.22}$$

For example, at $m = 2$, we have that the following integral tensor-reduces to

$$\begin{aligned}
&J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu] = \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \int \frac{\mathrm{d}^D K_1}{i\pi^{D/2}} \frac{\mathrm{d}^D K_2}{i\pi^{D/2}} \left(K_1^\mu - \frac{c}{a} K_2^\mu + X^\mu\right)\left(K_1^\nu - \frac{c}{a} K_2^\nu + X^\nu\right) \exp\left(\sum_i x_i A_i\right) \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \int \frac{\mathrm{d}^D K_1}{i\pi^{D/2}} \frac{\mathrm{d}^D K_2}{i\pi^{D/2}} \left[K_1^\mu K_1^\nu + \left(\frac{c}{a}\right)^2 K_2^\mu K_2^\nu + X^\mu X^\nu\right] \exp\left(aK_1^2 + \frac{P}{a} K_2^2 + \frac{Q}{P}\right) \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left\{ X^\mu X^\nu R^D + \left[\left(-\frac{1}{2a^{D/2+1}} g^{\mu\nu}\right)\left(\int \frac{\mathrm{d}^D K_2}{i\pi^{D/2}} e^{PK_2^2/a}\right)\right.\right. \\
&\quad \left.\left. + \left(\frac{c}{a}\right)^2 \left(-\frac{1}{2}\left(\frac{a}{P}\right)^{D/2+1} g^{\mu\nu}\right)\left(\int \frac{\mathrm{d}^D K_1}{i\pi^{D/2}} e^{aK_1^2}\right)\right] e^{Q/P} \right\} \\
&= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[X^\mu X^\nu R^D - \frac{1}{2}\left(\frac{1}{a} + \frac{c^2}{aP}\right) g^{\mu\nu} R^D\right] = \int_{\nu_1\dots\nu_n} \mathrm{D}x \left(X^\mu X^\nu - \frac{b}{2P} g^{\mu\nu}\right) R^D,
\end{aligned} \tag{5.23}$$

where again the integrals are performed as in the last section. Proceeding this way for every possible combination $[\dots k^\mu \dots]$, a table of tensor reduction rules can be obtained, where for reasons that will become clear later in this Chapter it is assumed $X^\mu = Y^\mu = 0$:

Tensor reduction rules at 2 loops [7, p.40] (for $X^\mu = Y^\mu = 0$):

$$\begin{aligned}
J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left(-\frac{b}{2P} g^{\mu\nu}\right) R^D, \\
J^D_{\nu_1\dots\nu_n}[k_1^\mu k_2^\nu] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left(\frac{c}{2P} g^{\mu\nu}\right) R^D, \\
J^D_{\nu_1\dots\nu_n}[k_2^\mu k_2^\nu] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left(-\frac{a}{2P} g^{\mu\nu}\right) R^D, \\
J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu k_1^\rho k_1^\sigma] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[\frac{b^2}{4P^2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right] R^D, \\
J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu k_1^\rho k_2^\sigma] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[-\frac{bc}{4P^2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right] R^D, \\
J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu k_2^\rho k_2^\sigma] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[\frac{ab}{4P^2} g^{\mu\nu}g^{\rho\sigma} + \frac{c^2}{4P^2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right] R^D, \\
J^D_{\nu_1\dots\nu_n}[k_1^\mu k_2^\nu k_2^\rho k_2^\sigma] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[-\frac{ac}{4P^2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right] R^D, \\
J^D_{\nu_1\dots\nu_n}[k_2^\mu k_2^\nu k_2^\rho k_2^\sigma] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[\frac{a^2}{4P^2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right] R^D, \\
&\dots
\end{aligned} \tag{5.24}$$

The same table can be straightforwardly obtained for 1 loop momentum integrals directly form (5.24) by equating (5.13) with (5.4) and by identifying their coefficients: it is clear that in order to obtain 1 loop rules from 2 loops ones we must set $b = 1$ and $c = 0$. This gives

Tensor reduction rules at 1 loop [7, p.42] (for $X^\mu = Y^\mu = 0$):

$$\begin{aligned} J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left(-\frac{1}{2P} g^{\mu\nu}\right) R^D, \\ J^D_{\nu_1\dots\nu_n}[k_1^\mu k_1^\nu k_1^\rho k_1^\sigma] &= \int_{\nu_1\dots\nu_n} \mathrm{D}x \left[\frac{1}{4P^2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right] R^D, \\ &\dots \end{aligned} \tag{5.25}$$

The original goal of this section was to link $J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots]$ to the scalar integral, however so far we've been only able to remove the indices from the loop momenta. We now need a way to link these newly obtained tensor-reduced integrals to scalar integrals. We can do that thanks to the following algebraic observation [7, p.39]:

$$\frac{(-1)^{\nu_i} x_i^{\nu_i - 1}}{\Gamma(\nu_i)} \left(\frac{x_i}{P}\right) R^D = (-\nu_i) \frac{(-1)^{\nu_i+1} x_i^{\nu_i}}{\Gamma(\nu_i + 1)} R^{D+2}. \tag{5.26}$$

The LHS is of great interest to us because the fraction could come from the measure $\mathrm{D}x$, the term in curly brackets could be part of an integrand and the R^D could be the last integrand bit of a tensor-reduced integral. By absorbing the x_i into $\mathrm{D}x$, increasing its respective power $\nu_i \to \nu_i + 1$ and producing a multiplicative factor of $-\nu_i$, and by absorbing the P into R^D, increasing its dimension $D \to D + 2$, we see that we can fully absorb all Schwinger representation coefficients and 4-vectors that end up in the integrand of the tensor-reduced integrals [7, p.39]: indeed they all are linear combinations of $x_1, \dots, x_n$, hence (5.26) can be systematically applied.

In conclusion, the tensor reduction procedure consists of two stages: in the first step (5.25) or (5.24) identities are used to bring general integrals $J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots]$ down to a tensor-reduced form, while in the second step the latter are brought into a linear combination of scalar integrals, each with various different $(\nu_1, \dots, \nu_n) \to (\nu'_1, \dots, \nu'_n)$ powers and different $D \to D'$ dimensions. Notice that the word linear combination is not an abuse of language here, because formally speaking what we call tensors are in fact their components.

5.1.4 Power reduction and dimensional shift

We'd now like to link scalar integrals with different powers and different dimensions among themselves. This is of great interest to us because, if we find out that all scalar integrals are eventually linked to a restricted set of **master integrals**, then we only need to analytically evaluate those and find transformation rules among scalar integrals to push all of them down to the master integrals.

Relationships between scalar integrals of same dimensions but different powers can be found by **integration by parts (IBP)**. They can be obtained by taking the partial derivatives [7, p.94]

$$0 = \int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D k_m}{i\pi^{D/2}} \left[\frac{\partial}{\partial a^\mu} \left(\frac{b^\mu}{A_1^{\nu_1} \cdots A_n^{\nu_n}}\right)\right], \tag{5.27}$$

where $a \in \{k_1, \dots, k_m\}$ and $b \in \{k_1, \dots, k_m, p_1, \dots, p_r\}$. This provides us $m \cdot [m + (r - 1)]$ distinct IBP identities [7, p.94], the $(r - 1)$ being that due to conservation of external

momentum, allowing us to perform the **power reduction** of the scalar integrals down to a linear combination of master integrals, all of the same dimension. Note that many of these IBP identities will be trivially null, especially those with $b \in \{p_1, \dots, p_r\}$, therefore in practice there will be less than $m \cdot [m + (r-1)]$ identities. Note also that, in certain pathological cases, the IBP identities are not enough to reduce all the scalar integrals, and therefore they must be complemented with additional identities obtained from Lorentz invariance [7, p.96].

Now that our original linear combination of scalar integrals of different powers and different dimensions has been reduced to a linear combination of master integrals of still different dimensions, we need to find relationships between master integrals of different dimensions. This can be done by observing that, for a given master integral of powers $(\nu_1, \dots, \nu_n)$ and dimension D, we have [7, p.97]

$$J^D_{\nu_1 \dots \nu_n} = \int_{\nu_1 \dots \nu_n} \mathrm{D}x \, \frac{e^{Q/P}}{P^{D/2}} = \int_{\nu_1 \dots \nu_n} \mathrm{D}x \, \frac{P e^{Q/P}}{P^{(D+2)/2}} = \int_{\nu_1 \dots \nu_n} \mathrm{D}x \, P R^{D+2} \tag{5.28}$$

and hence (5.26) can be applied, giving us once again a linear combination of scalar integrals of different powers but this time, crucially, all of dimension $D+2$ since no P denominators are present. If we now apply the power reduction to every scalar integral in the above-mentioned linear combination, at the end of the day we get an equivalence between the original master integral, of powers $(\nu_1, \dots, \nu_n)$ and dimension D, and a linear combination of all the others master integrals, all in dimension $D + 2$. Repeating the above observation for each master integral provides us with a system of equations linking all master integrals of dimension D to all master integrals of dimension $D + 2$. Inverting this system will therefore lead to the transformation rules from master integrals of dimension $D + 2$ to dimension D [7, p.98]. This **dimensional shift** is enough to systematically transform any master integral of any dimension $D' > D$ down to D, since P denominators in tensor-reduced integrals can only shift scalar integral dimensions by a minimal variation of 2.

5.2 Approach outline

We summarize the findings of the previous section to show the whole picture of the general approach under which our computer-algebra program will operate. The automated evaluation of integrals of the form (5.1) proceeds as follows [7, p.98]:

1. **Tensor reduction**: Apply rules (5.25) or (5.24) and absorb all Schwinger parameters according to (5.26) in order to obtain

$$J^D_{\nu_1 \dots \nu_n}[\dots k^\mu \dots] = \sum_i \alpha_i^{(\dots \mu \dots)} J^{D(i)}_{\nu_1(i) \dots \nu_n(i)}, \tag{5.29}$$

   where the coefficients of the linear combination $\alpha_i^{(\dots \mu \dots)} \in \mathbb{R}$ contain fixed tensors;

2. **Power reduction**: Apply the IBP identities to obtain

$$J^D_{\nu_1 \dots \nu_n}[\dots k^\mu \dots] = \sum_i \alpha_i^{(\dots \mu \dots)} \sum_j \beta_{ij} B_j^{D(i)}, \tag{5.30}$$

   where the coefficients of the linear combination are $\beta_{ij} \in \mathbb{R}$ and where we denote by $\{B_j^D\}$ the set of master integrals at a given dimension D;

3. **Dimensional shift**: Apply dimensional shift transformations to obtain

$$J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots] = \sum_i \alpha_i^{(\dots\mu\dots)} \sum_j \beta_{ij} \sum_k \gamma_{ijk} B^D_k, \tag{5.31}$$

where the coefficients of the linear combination are $\gamma_{ijk} \in \mathbb{R}$.

We therefore end up with

$$J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots] = \sum_{ijk} \left(\alpha_i^{(\dots\mu\dots)} \beta_{ij} \gamma_{ijk}\right) B^D_k, \tag{5.32}$$

and indeed only an explicit analytical evaluation of the master integrals $\{B^D_k\}$ is needed.

5.3 Application to 1 and 2 loops integrals

If there exists some $\chi_{il} \neq 0$ in (5.2) we use (4.10) to remove any external momenta p_l that may appear in the free propagators, as it was already mentioned in the previous Chapter 4. Indeed, this allows us to systematically approximate any loop integral into a combination of (5.1) integrals for which all coefficients $\chi_{il} = 0$.

In order to avoid complications that will be discussed in section 5.4, for the $m = 1$ loop case we'll consider $n = 1$ internal propagator of mass σ, while for the $m = 2$ loops case we'll consider up to $n = 3$ internal propagators of masses $\rho \neq \sigma \neq \tau \neq \rho$. Up to a constant $A \in \mathbb{R}$, the scalar integral of the former takes the form of (3.14), while the scalar integral of the latter takes the form

$$K^D_{\nu_1,\nu_2,\nu_3} := A^2 J^D_{\nu_1,\nu_2,\nu_3} = \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{1}{(k_1^2 - \rho^2)^{\nu_1}} \frac{1}{[(k_1 - k_2)^2 - \tau^2]^{\nu_2}} \frac{1}{(k_2^2 - \sigma^2)^{\nu_3}}. \tag{5.33}$$

Here we have defined new generic integrals $K^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots] := A^m J^D_{\nu_1\dots\nu_n}[\dots k^\mu \dots]$, hence new scalar integrals $K^D_{\nu_1\dots\nu_n} := A^m J^D_{\nu_1\dots\nu_n}$, via the proportionality constant $A = i/(2^D \pi^{D/2})$. With that, one might be tempted to directly apply all the J-derived formulas on the new K-integrals: while this can actually be done, one must be aware of the manipulations under the hood implied by the constant $A = A(D)$, which contains information about the dimension. Indeed, if all IBP identities (5.27) are left invariant by any constant multiplication, the same can't be said for the tensor reduction rules (5.25), (5.24) and the dimensional shifts (5.28). This is because if we absorb the constant $A(D)$ into a new $\tilde{R}^D := R^D A^m(D)$ to be used in the K-integrals representations, then for any P in their denominator we would get

$$\int_{\nu_1\dots\nu_n} \mathrm{D}x \left(\frac{1}{P}\right) \tilde{R}^D = \int_{\nu_1\dots\nu_n} \mathrm{D}x\, \tilde{R}^{D+2} \frac{A^m(D)}{A^m(D+2)} = (4\pi)^m \int_{\nu_1\dots\nu_n} \mathrm{D}x\, \tilde{R}^{D+2}. \tag{5.34}$$

However, this extra $(4\pi)^m$ factor that would be picked up when the dimension increases by 2, namely during tensor reduction, would be cancelled by an extra $(4\pi)^{-m}$ factor that would be picked up when the dimension decreases by 2, namely during dimensional shifts: globally, the two effects compensate. Another way to see this is to simply factor out the constant $A(D)$ right from the start of any calculation, proceed with J-integrals, and at the end convert everything back into K-integrals by absorbing $A(D)$. As a result, in this thesis, we'll abusively operate a $J \to K$ formal substitution on all J-formulas and we'll abusively refer to the new generic and the new scalar K-integrals as if they were the usual generic and

scalar J-integrals. Moreover, if all three masses are equal, we'll denote the scalar integrals as

$$I^D_{\nu_1,\nu_2,\nu_3} := \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{1}{(k_1^2-\sigma^2)^{\nu_1}} \frac{1}{[(k_1-k_2)^2-\sigma^2]^{\nu_2}} \frac{1}{(k_2^2-\sigma^2)^{\nu_3}}, \tag{5.35}$$

while if two of them are equal and one is different we'll denote the scalar integrals as

$$H^D_{\nu_1,\nu_2,\nu_3} := \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{1}{(k_1^2-\rho^2)^{\nu_1}} \frac{1}{[(k_1-k_2)^2-\sigma^2]^{\nu_2}} \frac{1}{(k_2^2-\sigma^2)^{\nu_3}}. \tag{5.36}$$

The above formal substitution will obviously also work for $K \to I, H$.

If for (3.14) a general analytical formula is known for all $\nu \in \mathbb{N}$, only the (3.35), (3.37) and (3.39) cases are known for $I^D_{\nu_1,\nu_2,\nu_3}$, $H^D_{\nu_1,\nu_2,\nu_3}$ and $K^D_{\nu_1,\nu_2,\nu_3}$: this is however not a problem at all for all previous integrals, even if we didn't have the general analytical formula for (3.14), because we know that thanks to the IBP identities we are able to go down to their respective basis of master integrals $\{B^D_i\}$. In this section we'll derive the sets of master integrals for the first two types of generic integrals, namely for the 1 loop T^D_ν and for the 2 loops symmetric $I^D_{\nu_1,\nu_2,\nu_3}$. An attempt to determine the master integrals for the 2 loops semi-symmetric integrals $H^D_{\nu_1,\nu_2,\nu_3}$ will also be made.

5.3.1 Index symmetries

1 loop

As there is only one index, no index symmetries are needed.

2 loops symmetric

The scalar integral is fully symmetric under any S_3 permutation of the indices

$$I^D_{\nu_1,\nu_2,\nu_3} = I^D_{\nu_3,\nu_2,\nu_1} = I^D_{\nu_2,\nu_1,\nu_3} = I^D_{\nu_1,\nu_3,\nu_2}, \tag{5.37}$$

where the first equality is obtained by simply $k_1 \leftrightarrow k_2$, while the other two are obtained through the successive change of variables $k_{1,2} \to k_{1,2} + k_{2,1}$ and $k_{2,1} \to -k_{2,1}$ (the comma separates the two cases).

2 loops semi-symmetric

Because there's one different mass the first index can't be permuted, therefore we only have one symmetry

$$H^D_{\nu_1,\nu_2,\nu_3} = H^D_{\nu_1,\nu_3,\nu_2}, \tag{5.38}$$

obtained after successively sending $k_2 \to k_2 + k_1$ and then $k_1 \to -k_1$.

5.3.2 Schwinger representation coefficients

1 loop

Since (at least up to rank 4) there are no (5.12) coefficients appearing in (5.25), there's no need to derive them, but for the sake of completeness we give them anyway:

$$a = x, \qquad d^\mu = 0, \qquad f = -\sigma x, \qquad X^\mu = 0, \qquad P = x, \qquad Q = fP. \tag{5.39}$$

2 loops symmetric

Because it is $n = 3$, the Schwinger parameters are $x_1, x_2, x_3 \in \mathbb{R}$ and the coefficients and 4-vectors appearing in the Schwinger sum (5.3) are trivially determined as

$$\begin{gathered} a = x_{12}, \qquad b = x_{23}, \qquad c = -x_2, \qquad d^\mu = 0, \qquad e^\mu = 0, \qquad f = -\sigma^2 x_{123}, \\ X^\mu = 0, \qquad Y^\mu = 0, \qquad P = x_1x_2 + x_1x_3 + x_2x_3, \qquad Q = fP. \end{gathered} \tag{5.40}$$

2 loops semi-symmetric

The coefficients are identical to the above, with the exception of $f = -x_1\rho^2 - x_{23}\sigma^2$.

5.3.3 Deriving the tensor reduction formulas

Because (4.10) is applied we have that all $\chi_{il} = 0$. This means that the Schwinger 4-vectors we obtain in the sums (5.12) and (5.3) are set to $d^\mu = e^\mu = 0$, ultimately leading in both cases to $X^\mu = Y^\mu = 0$. Therefore (5.25) and (5.24) in their given form can be directly applied. Also, we notice that all tensor reduction rules (5.25) and (5.24) are null for an odd number of tensor indices.

5.3.4 Deriving the power reduction formulas

1 loop

There's just one useful (from now on this will be implicit) IBP identity at $m = 1$, namely

$$0 = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - \sigma^2)^\nu} = (D - 2\nu)T^D_\nu - 2\nu\sigma^2 T^D_{\nu+1}, \tag{5.41}$$

implying the power reduction formula

$$T^D_\nu = \frac{D - 2(\nu - 1)}{2(\nu - 1)\sigma^2} T^D_{\nu-1}. \tag{5.42}$$

From the above we deduce that at one loop there's only one master integral and that is

1 loop master integrals:

$$\{B^D_j\} = \{T^D_1\}. \tag{5.43}$$

2 loops symmetric

There are a total of 4 IBP identities. The first two of them are derived by integrating the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial k_1^\mu}\left(\frac{k_1^\mu}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}}\right) &= \frac{D}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}} + k_1^\mu\left(-\nu_1\frac{2k_{1,\mu}}{A_1^{\nu_1+1}A_2^{\nu_2}A_3^{\nu_3}} - \nu_2\frac{2(k_{1,\mu} - k_{2,\mu})}{A_1^{\nu_1}A_2^{\nu_2+1}A_3^{\nu_3}}\right) \\ &= \frac{D}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}} - 2\nu_1\frac{(k_1^2 - \sigma^2) + \sigma^2}{A_1^{\nu_1+1}A_2^{\nu_2}A_3^{\nu_3}} \\ &\quad - \nu_2\frac{(k_1^2 - 2k_1\cdot k_2 + k_2^2 - \sigma^2) + (k_1^2 - \sigma^2) - (k_2^2 - \sigma^2) + \sigma^2}{A_1^{\nu_1}A_2^{\nu_2+1}A_3^{\nu_3}} \end{aligned} \tag{5.44}$$

and the similarly computed

$$\begin{aligned}\frac{\partial}{\partial k_1^\mu}\left(\frac{k_2^\mu}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}}\right) &= \nu_1\frac{(k_1^2-2k_1\cdot k_2+k_2^2-\sigma^2)-(k_1^2-\sigma^2)-(k_2^2-\sigma^2)-\sigma^2}{A_1^{\nu_1+1}A_2^{\nu_2}A_3^{\nu_3}}\\ &+\nu_2\frac{(k_1^2-2k_1\cdot k_2+k_2^2-\sigma^2)+(k_2^2-\sigma^2)-(k_1^2-\sigma^2)+\sigma^2}{A_1^{\nu_1}A_2^{\nu_2+1}A_3^{\nu_3}},\end{aligned} \tag{5.45}$$

so that after integration one gets

$$\begin{aligned}0 &= DI^D_{\nu_1,\nu_2,\nu_3}-2\nu_1 I^D_{\nu_1,\nu_2,\nu_3}-2\nu_1\sigma^2 I^D_{\nu_1+1,\nu_2,\nu_3}\\ &-\nu_2 I^D_{\nu_1,\nu_2,\nu_3}-\nu_2 I^D_{\nu_1-1,\nu_2+1,\nu_3}+\nu_2 I^D_{\nu_1,\nu_2+1,\nu_3-1}-\nu_2\sigma^2 I^D_{\nu_1,\nu_2+1,\nu_3}\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}0 &= \nu_1 I^D_{\nu_1+1,\nu_2-1,\nu_3}-\nu_1 I^D_{\nu_1,\nu_2,\nu_3}-\nu_1 I^D_{\nu_1+1,\nu_2,\nu_3-1}-\nu_1\sigma^2 I^D_{\nu_1+1,\nu_2,\nu_3}\\ &+\nu_2 I^D_{\nu_1,\nu_2,\nu_3}+\nu_2 I^D_{\nu_1,\nu_2+1,\nu_3-1}-\nu_2 I^D_{\nu_1-1,\nu_2+1,\nu_3}+\nu_2\sigma^2 I^D_{\nu_1,\nu_2+1,\nu_3}.\end{aligned} \tag{5.47}$$

The last two IBP identities are derived by applying the swap $k_1 \leftrightarrow k_2$, hence $\nu_1 \leftrightarrow \nu_3$, to the previous two relations, giving us a total of 4 IBP identities

$$\begin{aligned}0 &= (D-2\nu_1-\nu_2)I^D_{\nu_1,\nu_2,\nu_3}-2\nu_1\sigma^2 I^D_{\nu_1+1,\nu_2,\nu_3}\\ &-\nu_2 I^D_{\nu_1-1,\nu_2+1,\nu_3}+\nu_2 I^D_{\nu_1,\nu_2+1,\nu_3-1}-\nu_2\sigma^2 I^D_{\nu_1,\nu_2+1,\nu_3},\end{aligned} \tag{5.48}$$

$$\begin{aligned}0 &= (D-2\nu_3-\nu_2)I^D_{\nu_1,\nu_2,\nu_3}-2\nu_3\sigma^2 I^D_{\nu_1,\nu_2,\nu_3+1}\\ &-\nu_2 I^D_{\nu_1,\nu_2+1,\nu_3-1}+\nu_2 I^D_{\nu_1-1,\nu_2+1,\nu_3}-\nu_2\sigma^2 I^D_{\nu_1,\nu_2+1,\nu_3},\end{aligned} \tag{5.49}$$

$$\begin{aligned}0 &= (\nu_2-\nu_1)I^D_{\nu_1,\nu_2,\nu_3}+\nu_1\big(I^D_{\nu_1+1,\nu_2-1,\nu_3}-I^D_{\nu_1+1,\nu_2,\nu_3-1}-\sigma^2 I^D_{\nu_1+1,\nu_2,\nu_3}\big)\\ &+\nu_2\big(I^D_{\nu_1,\nu_2+1,\nu_3-1}-I^D_{\nu_1-1,\nu_2+1,\nu_3}+\sigma^2 I^D_{\nu_1,\nu_2+1,\nu_3}\big),\end{aligned} \tag{5.50}$$

$$\begin{aligned}0 &= (\nu_2-\nu_3)I^D_{\nu_1,\nu_2,\nu_3}+\nu_3\big(I^D_{\nu_1,\nu_2-1,\nu_3+1}-I^D_{\nu_1-1,\nu_2,\nu_3+1}-\sigma^2 I^D_{\nu_1,\nu_2,\nu_3+1}\big)\\ &+\nu_2\big(I^D_{\nu_1-1,\nu_2+1,\nu_3}-I^D_{\nu_1,\nu_2+1,\nu_3-1}+\sigma^2 I^D_{\nu_1,\nu_2+1,\nu_3}\big).\end{aligned} \tag{5.51}$$

We notice that in all the above identities the sum of the indices of all the scalar integrals is either incremented by 0 or 1 with respect to $\nu_1+\nu_2+\nu_3$. In particular, (5.49) and (5.51) have their last terms in common, with a minus sign difference, therefore by combining them in a sum we obtain

$$\begin{aligned}0 &= (D-3\nu_3)I^D_{\nu_1,\nu_2,\nu_3}-3\nu_3\sigma^2 I^D_{\nu_1,\nu_2,\nu_3+1}+\nu_3\big(I^D_{\nu_1,\nu_2-1,\nu_3+1}-I^D_{\nu_1-1,\nu_2,\nu_3+1}\big)\\ &+2\nu_2\big(I^D_{\nu_1-1,\nu_2+1,\nu_3}-I^D_{\nu_1,\nu_2+1,\nu_3-1}\big),\end{aligned} \tag{5.52}$$

which can be cast into

$$\begin{aligned}I^D_{\nu_1,\nu_2,\nu_3+1} &= \frac{1}{3\nu_3\sigma^2}\Big[(D-3\nu_3)I^D_{\nu_1,\nu_2,\nu_3}+2\nu_2\big(I^D_{\nu_1-1,\nu_2+1,\nu_3}-I^D_{\nu_1,\nu_2+1,\nu_3-1}\big)\\ &+\nu_3\big(I^D_{\nu_1,\nu_2-1,\nu_3+1}-I^D_{\nu_1-1,\nu_2,\nu_3+1}\big)\Big],\end{aligned} \tag{5.53}$$

eventually giving us the first power reduction formula

$$I^D_{\nu_1,\nu_2,\nu_3} = \frac{1}{3(\nu_3-1)\sigma^2}\Big[(D-3\nu_3+3)I^D_{\nu_1,\nu_2,\nu_3-1} + 2\nu_2\big(I^D_{\nu_1-1,\nu_2+1,\nu_3-1} - I^D_{\nu_1,\nu_2+1,\nu_3-2}\big) + (\nu_3-1)\big(I^D_{\nu_1,\nu_2-1,\nu_3} - I^D_{\nu_1-1,\nu_2,\nu_3}\big)\Big]. \tag{5.54}$$

The above is indeed a power reduction because, even if locally some indices increment instead of decreasing, overall the sum of the indices on the RHS is lower by 1 than the sum of the LHS indices. Therefore, if after every application of (5.54) we reorder all indices with (5.37) from the lowest one to the highest one, then after a finite amount of steps either all indices will end up being equal to $\nu_1 = \nu_2 = \nu_3 = 1$ or one of them will singularly hit $\nu_1 = 0$ before the others. In the former case the power reduction stops at I^D_{111} because the denominator in (5.54) requires $\nu_3 > 1$. In the latter case we switch to a second power reduction formula

$$I^D_{0,\mu,\nu} = \frac{D-2(\nu-1)}{2(\nu-1)\sigma^2} I^D_{0,\mu,\nu-1}, \tag{5.55}$$

obtained from (5.48) by setting $\nu_2 = 0$. Notice that this is nothing more than an application of (5.42), because the integral can be split into $I^D_{0,\mu,\nu} = T^D_\mu T^D_\nu$. We clearly see that, provided the highest index $\nu > 1$, we can spiral down to I^D_{011}. In conclusion, with our two power reduction formulas (5.54) and (5.55), eventually we'll reach either I^D_{111} or I^D_{011}, which therefore constitute our master integrals at two loops

2 loops master integrals:

$$\big\{B^D_j\big\} = \big\{I^D_{011}, I^D_{111}\big\}. \tag{5.56}$$

As final remarks, notice that trivially $I^D_{000} = I^D_{001} = 0$, that $m = 2$ integrals require 2 integrations therefore $I^D_{001} \neq T^D_1$, and that the second $m = 2$ master integral can be computed from (3.20) as

$$I^D_{011} = (T^D_1)^2 = -\sigma^4 N^2\left(\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon^0)\right). \tag{5.57}$$

2 loops semi-symmetric

By proceeding analogously to the previous case, we find there are only 4 IBP identities

$$0 = (D-2\nu_1-\nu_2)H^D_{\nu_1,\nu_2,\nu_3} - 2\nu_1\rho^2 H^D_{\nu_1+1,\nu_2,\nu_3} - \nu_2 H^D_{\nu_1-1,\nu_2+1,\nu_3} + \nu_2 H^D_{\nu_1,\nu_2+1,\nu_3-1} - \nu_2\rho^2 H^D_{\nu_1,\nu_2+1,\nu_3}, \tag{5.58}$$

$$0 = (D-2\nu_3-\nu_2)H^D_{\nu_1,\nu_2,\nu_3} - 2\nu_3\sigma^2 H^D_{\nu_1,\nu_2,\nu_3+1} - \nu_2 H^D_{\nu_1,\nu_2+1,\nu_3-1} + \nu_2 H^D_{\nu_1-1,\nu_2+1,\nu_3} - \nu_2(2\sigma^2-\rho^2)H^D_{\nu_1,\nu_2+1,\nu_3}, \tag{5.59}$$

$$0 = (\nu_2-\nu_1)H^D_{\nu_1,\nu_2,\nu_3} + \nu_1\big(H^D_{\nu_1+1,\nu_2-1,\nu_3} - H^D_{\nu_1+1,\nu_2,\nu_3-1} - \rho^2 H^D_{\nu_1+1,\nu_2,\nu_3}\big) + \nu_2\big(H^D_{\nu_1,\nu_2+1,\nu_3-1} - H^D_{\nu_1-1,\nu_2+1,\nu_3} + (2\sigma^2-\rho^2)H^D_{\nu_1,\nu_2+1,\nu_3}\big), \tag{5.60}$$

$$0 = (\nu_2-\nu_3)H^D_{\nu_1,\nu_2,\nu_3} + \nu_3\big(H^D_{\nu_1,\nu_2-1,\nu_3+1} - H^D_{\nu_1-1,\nu_2,\nu_3+1} - \rho^2 H^D_{\nu_1,\nu_2,\nu_3+1}\big) + \nu_2\big(H^D_{\nu_1-1,\nu_2+1,\nu_3} - H^D_{\nu_1,\nu_2+1,\nu_3-1} + \rho^2 H^D_{\nu_1,\nu_2+1,\nu_3}\big), \tag{5.61}$$

where as a check we can set $\rho = \sigma$ to readily recover (5.48), (5.49), (5.50) and (5.51). Using the same argument as before, by subtracting two times (5.60) from (5.58) one gets

$$\begin{aligned} 0 = (D-3\nu_2)H^D_{\nu_1,\nu_2,\nu_3} + 2\nu_1\big(H^D_{\nu_1+1,\nu_2,\nu_3-1} - H^D_{\nu_1+1,\nu_2-1,\nu_3}\big) \\ + \nu_2\big(H^D_{\nu_1-1,\nu_2+1,\nu_3} - H^D_{\nu_1,\nu_2+1,\nu_3-1} + (\rho^2-4\sigma^2)H^D_{\nu_1,\nu_2+1,\nu_3}\big), \end{aligned} \tag{5.62}$$

which can be recast into

$$\begin{aligned} H^D_{\nu_1,\nu_2+1,\nu_3} = \frac{1}{\nu_2(4\sigma^2-\rho^2)}\Bigg[(D-3\nu_2)H^D_{\nu_1,\nu_2,\nu_3} \\ + 2\nu_1\big(H^D_{\nu_1+1,\nu_2,\nu_3-1} - H^D_{\nu_1+1,\nu_2-1,\nu_3}\big) + \nu_2\big(H^D_{\nu_1-1,\nu_2+1,\nu_3} - H^D_{\nu_1,\nu_2+1,\nu_3-1}\big)\Bigg], \end{aligned} \tag{5.63}$$

eventually giving us the power reduction formula

$$\begin{aligned} H^D_{\nu_1,\nu_2,\nu_3} = \frac{1}{(\nu_2-1)(4\sigma^2-\rho^2)}\Bigg[(D-3(\nu_2-1))H^D_{\nu_1,\nu_2-1,\nu_3} \\ + 2\nu_1\big(H^D_{\nu_1+1,\nu_2-1,\nu_3-1} - H^D_{\nu_1+1,\nu_2-2,\nu_3}\big) + (\nu_2-1)\big(H^D_{\nu_1-1,\nu_2,\nu_3} - H^D_{\nu_1,\nu_2,\nu_3-1}\big)\Bigg]. \end{aligned} \tag{5.64}$$

Again, we can check the result by setting $\rho = \sigma$ to readily obtain (5.54). Sadly, this marks the end of our attempt to find the set of needed IBP relations for the 2 loops semi-symmetric case. If we use (5.38) to order the last two indices from lowest to highest, the first scalar integral of the above will end up in a $H^D_{\nu_1,1,1}$ form, while all the others will end up in a $H^D_{\nu_1,0,1}$, $H^D_{\nu_1,0,\nu_3}$ or H^D_{0,ν_2,ν_3} form. While the latter cases, containing a null index, can be easily reduced to H^D_{101} and H^D_{011} by dividing the integrals into a multiplication of tadpoles (with same or different masses) and by using the usual (5.42), we couldn't find an analytical IBP formula to reduce the former cases to, we expect, H^D_{111}. Indeed, no one guaranteed us that such a formula exists, and even more so no one told us that the power reduction had to be done that way, that is, it could very well be that $H^D_{\nu_1,1,1}$ integrals simply can't be directly reduced to H^D_{111} and one should avoid to get them in the first place. For these reasons, instead of directly implementing analytical IBP formulas in our computer-algebra code as we did, we could have generated IBP tables instead. Starting from the lowest possible indices, we would have generated them by repeatedly applying the 4 IBP identities to link the higher-indices scalar integrals to the starting ones, and so on, such that the power reduction descent would continuously lookup for an appropriate path in said tables, without the need to determine specific analytical formulas. Another possibility would have been to use the alternative method presented in the upcoming section 5.4.

5.3.5 Useful 1 and 2 loops scalar integrals

We now apply the above freshly derived $m = 1$ and $m = 2$ IBP identities to pre-calculate a list of useful scalar integrals for later use in the thesis. At $m = 1$ loop, from (5.42), we get

$$T^D_4 = \frac{(D-6)(D-4)(D-2)}{48\sigma^6}T^D_1, \tag{5.65}$$

$$T^D_3 = \frac{(D-4)(D-2)}{8\sigma^4}T^D_1, \tag{5.66}$$

$$T^D_2 = \frac{D-2}{2\sigma^2}T^D_1. \tag{5.67}$$

At $m = 2$ loops, from (5.55), we can derive

$$I^D_{022} = \frac{D-2}{2\sigma^2} I^D_{012} = \left(\frac{D-2}{2\sigma^2}\right)^2 I^D_{011}, \tag{5.68}$$

$$I^D_{023} = \frac{D-2}{2\sigma^2} I^D_{013} = \frac{(D-4)(D-2)^2}{8\sigma^6} I^D_{011}, \tag{5.69}$$

and from (5.48) we can obtain a multitude of scalar integrals

$$I^D_{112} = \frac{D-3}{3\sigma^2} I^D_{111}, \tag{5.70}$$

$$I^D_{113} = \frac{1}{6\sigma^2}\left[\frac{(D-8)(D-3)}{3\sigma^2} I^D_{111} + 2\left(\frac{D-2}{2\sigma^2}\right)^2 I^D_{011}\right], \tag{5.71}$$

$$I^D_{122} = \frac{1}{3\sigma^2}\left[(D-2) I^D_{112} - I^D_{022}\right] = \frac{1}{3\sigma^2}\left[\frac{(D-3)(D-2)}{3\sigma^2} I^D_{111} - \left(\frac{D-2}{2\sigma^2}\right)^2 I^D_{011}\right], \tag{5.72}$$

$$I^D_{123} = \frac{2}{3}\left[\frac{D-6}{4\sigma^2} I^D_{122} + \frac{1}{2\sigma^2}\left(I^D_{023} - I^D_{113}\right)\right]. \tag{5.73}$$

5.3.6 Deriving the dimensional shift formulas

1 loop

Using the $m = 1$ Schwinger representation coefficients we obtain

$$T^D_1 = \int_1 \mathrm{D}x \, \frac{P e^{Q/P}}{P^{(D+2)/2}} = -T^{D+2}_2 = -\frac{D}{2\sigma^2} T^{D+2}_1. \tag{5.74}$$

Notice the minus sign coming from the multiplicative factor $-\nu$, with $\nu = 1$. This immediately gives us the following result which, if expanded in ε, gives

$$T^{D+2}_1 = -\frac{2\sigma^2}{D} T^D_1 = -\frac{\sigma^2}{2}\left(1 + \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)\right) T^D_1. \tag{5.75}$$

The successive application of the above results yields

$$T^{D+4}_1 = \frac{4\sigma^4}{(D+2)D} T^D_1 = \frac{\sigma^4}{6}\left(1 + \frac{5\varepsilon}{6} + \mathcal{O}(\varepsilon^2)\right) T^D_1. \tag{5.76}$$

2 loops symmetric

Using the $m = 2$ Schwinger representation coefficients we obtain

$$\begin{aligned} I^D_{011} &= \int_{011} \mathrm{D}x \, \frac{P e^{Q/P}}{P^{(D+2)/2}} = I^{D+2}_{022} = \frac{D^2}{4\sigma^4} I^{D+2}_{011}, \\ I^D_{111} &= \int_{111} \mathrm{D}x \, \frac{P e^{Q/P}}{P^{(D+2)/2}} = 3 I^{D+2}_{122} = \frac{D(D-1)}{3\sigma^4} I^{D+2}_{111} - \frac{D^2}{4\sigma^6} I^{D+2}_{011}, \end{aligned} \tag{5.77}$$

where in both third equalities we used the pre-calculated results from section 5.3.5. By inverting the above system of equations and by expanding everything in ε one gets

$$
\begin{aligned}
I^{D+2}_{011} &= \frac{4\sigma^4}{D^2} I^D_{011} = \frac{\sigma^4}{4}\big(1+\varepsilon+\mathcal{O}(\varepsilon^2)\big) I^D_{011}, \\
I^{D+2}_{111} &= \frac{3\sigma^4}{D(D-1)}\left[\frac{I^D_{011}}{\sigma^2} + I^D_{111}\right] = \frac{\sigma^4}{4}\left(1+\frac{7\varepsilon}{6}+\mathcal{O}(\varepsilon^2)\right)\left[\frac{I^D_{011}}{\sigma^2} + I^D_{111}\right].
\end{aligned}
\tag{5.78}
$$

5.4 Alternative methods

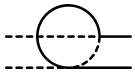
5.4.1 Complications

Why did we only allow $n = 1$ internal propagators in the $m = 1$ case and up to $n = 3$ internal propagators in the $m = 2$ case? Consider an interaction vertex among two scalar fields ϕ_1 and ϕ_2 with, crucially, different masses $m_1 \neq m_2$. If we introduce


$$:= -iK_4, \tag{5.79}$$

where the straight line denotes ϕ_1 and the dashed one ϕ_2, then the following 1 loop and 2 loops diagrams are allowed


$$\sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k+p)^2 - m_1^2} \frac{1}{k^2 - m_2^2},$$


$$\sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{k^2 - m_1^2} \frac{1}{(k+p)^2 - m_2^2} \frac{1}{(k-l)^2 - m_2^2} \frac{1}{(l+q)^2 - m_1^2}, \tag{5.80}$$

where p, q are generic (combinations of) external momenta. We see that the first diagram requires $n = 2$ internal propagators, while the second one requires $n = 4$ propagators, so these diagrams can't be analytically computed by our computer-algebra program. In this section we show how we tried to implement the treatment of these diagrams in our code, without success, leading to their exclusion.

5.4.2 Handling of 1 loop diagrams

To allow for two propagators of different masses in 1 loop diagrams, one could have defined a two-indices tadpole

$$T^D_{\mu,\nu} := \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \rho^2)^\mu} \frac{1}{(k^2 - \sigma^2)^\nu}, \tag{5.81}$$

which wouldn't get confused with the standard one because it has two indices, indicating the presence of two different masses $\rho \neq \sigma$. Clearly, if it is set $\rho = \sigma$ instead, we recover the usual $T^D_{\mu+\nu}$ tadpole (3.14). Then, 4 IBP identities would have been derived

$$0 = DT^D_{\mu,\nu} - 2\mu\big(T^D_{\mu,\nu} + \rho^2 T^D_{\mu+1,\nu}\big) - 2\nu\big(T^D_{\mu,\nu} + \sigma^2 T^D_{\mu,\nu+1}\big), \tag{5.82}$$

$$0 = DT^D_{\mu,\nu} - 2\mu\big(T^D_{\mu+1,\nu-1} + \sigma^2 T^D_{\mu+1,\nu}\big) - 2\nu\big(T^D_{\mu-1,\nu+1} + \rho^2 T^D_{\mu,\nu+1}\big), \tag{5.83}$$

$$0 = DT^D_{\mu,\nu} - 2\mu\big(T^D_{\mu,\nu} + \rho^2 T^D_{\mu+1,\nu}\big) - 2\nu\big(T^D_{\mu-1,\nu+1} + \rho^2 T^D_{\mu,\nu+1}\big), \tag{5.84}$$

$$0 = DT^D_{\mu,\nu} - 2\mu\big(T^D_{\mu+1,\nu-1} + \sigma^2 T^D_{\mu+1,\nu}\big) - 2\nu\big(T^D_{\mu,\nu} + \sigma^2 T^D_{\mu,\nu+1}\big), \tag{5.85}$$

and, subtracting (5.84) from (5.82), one would have obtained

$$0 = 2\nu\big(T^D_{\mu-1,\nu+1} + \rho^2 T^D_{\mu,\nu+1} - T^D_{\mu,\nu} - \sigma^2 T^D_{\mu,\nu+1}\big), \tag{5.86}$$

meaning that, for any $\nu \neq 0$, the power reduction formula

$$T^D_{\mu,\nu+1} = \frac{T^D_{\mu-1,\nu+1} - T^D_{\mu,\nu}}{\sigma^2 - \rho^2} \tag{5.87}$$

could have been used to obtain scalar integrals of either the form $T^D_{\mu,0} = T^D_{\mu}\big|_{\sigma=\rho}$ or $T^D_{0,\nu} = T^D_{\nu}$. It turns out that the formula also works for the $\nu = 0$ edge case because, more directly,

$$\begin{aligned} T^D_{\mu,\nu+1} &= \frac{1}{\sigma^2-\rho^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\sigma^2-\rho^2}{(k^2-\rho^2)^\mu (k^2-\sigma^2)^{\nu+1}} \\ &= \frac{1}{\sigma^2-\rho^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(k^2-\rho^2)-(k^2-\sigma^2)}{(k^2-\rho^2)^\mu (k^2-\sigma^2)^{\nu+1}} = \frac{T^D_{\mu-1,\nu+1} - T^D_{\mu,\nu}}{\sigma^2-\rho^2}. \end{aligned} \tag{5.88}$$

Unfortunately, the attempted implementation of the above power reduction formula led to momentum-dependent divergences, which shouldn't appear because the momentum-dependent contributions arising from (4.10) are UV-convergent. This might be due to the fact that we are in presence of different masses tadpoles, spoiling their cancellation in (4.10).

5.4.3 Handling of 2 loops diagrams

Because we derived all our results with at most $n = 3$ internal propagators, the treatment of diagrams with $n > 3$ propagators would require a complete resumption of the framework presented in this chapter. But even at $n = 3$ we encountered issues, namely we weren't able to power reduce $H^D_{\nu_1,\nu_2,\nu_3}$ integrals, let alone $K^D_{\nu_1,\nu_2,\nu_3}$ ones. A possibility to bypass this difficulty to determine the correct analytical IBP identities would be to consider said integrals from a different perspective.

Define the position vector $\mathbf{r} := (r_1, r_2, r_3)$, the mass vector $\mathbf{m} := (\rho, \sigma, \tau)$ and reinterpret the general sunset as

$$K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{r}) := \int \frac{\mathrm{d}^d k}{(2\pi)^4}\frac{\mathrm{d}^d l}{(2\pi)^4}\frac{1}{(k^2-r_1^2)^{\nu_1}}\frac{1}{(l^2-r_2^2)^{\nu_2}}\frac{1}{[(k-l)^2-r_3^2]^{\nu_3}}. \tag{5.89}$$

It is then obvious that

$$K^D_{\nu_1,\nu_2,\nu_3} = K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{m}) = K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{m}}, \tag{5.90}$$

so, as long as at the end we evaluate $K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{r})$ at $\mathbf{r} = \mathbf{m}$, we can apply whatever operation we wish to $K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{r})$ in order to modify it as we like. For instance, we would like to manipulate the power of the propagators. An application of a partial derivative can do that, namely

$$\frac{\partial}{\partial(r_1^2)}K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{r}) = \nu_1 K^D_{\nu_1+1,\nu_2,\nu_3}(\mathbf{r}) \quad \Rightarrow \quad K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{r}) = \frac{1}{\nu_1-1}\frac{\partial}{\partial(r_1^2)}K^D_{\nu_1-1,\nu_2,\nu_3}(\mathbf{r}), \tag{5.91}$$

where in this example we considered the A_1 propagator associated to the ν_1 power. With that idea at hand, it is now easy to obtain the general result

$$K^D_{\nu_1,\nu_2,\nu_3}(\mathbf{m}) = \left[\frac{\partial^{\nu_1-1}_{(r_1^2)}\partial^{\nu_2-1}_{(r_2^2)}\partial^{\nu_3-1}_{(r_3^2)}}{(\nu_1-1)!(\nu_2-1)!(\nu_3-1)!}K^D_{111}(\mathbf{r})\right]_{\mathbf{r}=\mathbf{m}}, \tag{5.92}$$

which would allow to systematically reduce any general $\nu_1, \nu_2, \nu_3 > 0$ sunset down to the K^D_{111} master integral, as long as one is willing to compute all the $\nu_1 + \nu_2 + \nu_3 - 3$ derivatives. The latter might even become a trivial task should K^D_{111} be simple enough, as in (3.39). Notice that, crucially, the $\mathbf{r} = \mathbf{m}$ evaluation takes place after the partial derivatives acted on the integral. If this seems obvious for the general $\rho \neq \sigma \neq \tau \neq \rho$ case, it is less so in the cases where two or all three masses are equal: one must keep working in the general setting with three different independent $\mathbf{r}$ coordinates, and only at the end substitute with the mass vector $\mathbf{m}$. The main downside of (5.92) is that it makes dimensional shifts difficult, because under this setting one has to essentially rederive the general sunset, starting from (3.22) and refraining from expanding in ε. A Fourier transformation of (5.92) was attempted, and a formal dimensional shift integral formula was obtained, but its evaluation is certainly more challenging than the previous suggestion.

To verify the veracity of this method, consider the (3.39) reinterpretation

$$K^D_{111} = -\frac{1}{2}N^2(r_2)\left\{\frac{\mathbf{r}^2}{\varepsilon^2} + \frac{1}{\varepsilon}\left[3\mathbf{r}^2 - 2r_1^2 \ln\left(\frac{r_1}{r_2}\right)^2 - 2r_3^2 \ln\left(\frac{r_3}{r_2}\right)^2\right] + \mathcal{O}\big(\varepsilon^0\big)\right\}. \tag{5.93}$$

First, we check the section 5.3.5 result

$$I^D_{112} = \frac{D-3}{3\sigma^2} I^D_{111} = -\frac{1}{2}N^2\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + \mathcal{O}\big(\varepsilon^0\big)\right). \tag{5.94}$$

To do so, select the mass vector $\mathbf{m} = (\sigma, \sigma, \sigma)$ and apply (5.92) to readily obtain

$$\begin{aligned} I^D_{112} = K^D_{112}(\mathbf{m}) &= \Big[\partial_{(r_3^2)} K^D_{111}(\mathbf{r})\Big]_{\mathbf{r}=\mathbf{m}} \\ &= \left[-\frac{1}{2}N^2(r_2)\left\{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[3 - 2\ln\left(\frac{r_3}{r_2}\right)^2 - 2\right] + \mathcal{O}\big(\varepsilon^0\big)\right\}\right]_{\mathbf{r}=\mathbf{m}} \\ &= -\frac{1}{2}N^2\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + \mathcal{O}\big(\varepsilon^0\big)\right). \end{aligned} \tag{5.95}$$

As a second check, we want to recover the dimensional shift (5.75) result. Using (3.11) and the notable $\Gamma(z+1) = z\Gamma(z)$ property of the Gamma function, one obtains

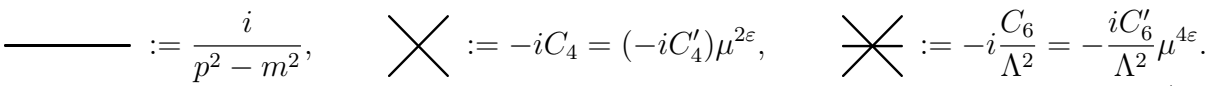
$$T^{D+2n}_{\nu} = \int \frac{\mathrm{d}^{d+2n}k}{(2\pi)^{d+2n}} \frac{1}{(k^2-\sigma^2)^{\nu}} = \frac{\sigma^{2n}}{(4\pi)^n} \frac{T^D_{\nu}}{\left(\nu - \frac{d}{2} - 1\right)\cdots\left(\nu - \frac{d}{2} - n\right)}, \tag{5.96}$$

with the correct $(4\pi)^{-n}$ extra factor. If one now sets $n = 1$ and $\nu = 1$, (5.75) is immediately obtained. Setting $n = 2$ and $\nu = 1$ instead gives the double successive application of (5.75), that is (5.76).

Chapter 6

Analytical results of diagrams

In this Chapter we run the whole symbolic evaluation procedure discussed in the previous Chapter 5 to obtain the amplitudes of all 1 loop and 2 loops 1PI diagrams appearing in the 1PI n-point functions of a theory composed of a single scalar field of mass m and whose Feynman rules are



(6.1)

The reason behind these rules will become clear in the upcoming Chapter 7, along with the explanation of why we consider $n \in \{2, 4, 6\}$. Up to the snowman diagram we analytically perform all the algorithm's steps, then we just give the computer-algebra computed result. Also, notice that all 1 loop results will provide a finite part, while all 2 loops ones won't: other than being a consequence of (3.20) and (3.35), this won't really be a problem because the renormalization of the 2 loops divergences can't occur through 2 loops finite parts, as they are already of 2 loops order. However, 1 loop finite parts could get multiplied by diverging terms of the renormalization constants, and as a result they might provide divergences of 2 loops order. Therefore, 1 loop finite parts must always be considered. Before giving the results, we need to discuss two more points.

6.1 Loops counting

The number of loops L of a Feynman diagram can be straightforwardly determined if one has access to a graphical representation of the diagram. But what if one has only access to the analytical value of its amplitude? In this section, we show that L can be determined by only knowing which coupling constants appear in amplitudes, and reciprocally we show how contributions of n-point functions can be classified by their loop order.

Consider an n-point function, with $n = n_B + n_F$ the number of, respectively, bosons and fermions. It admits two representations, namely one in position space and, by taking a Fourier transformation, one in momentum space

$$\begin{aligned} &G^{(n)}_{f_1\dots f_n}(x_1,\dots,x_n) = \langle f_1(x_1)\dots f_n(x_n)\rangle = \\ &= \int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\cdots\frac{\mathrm{d}^d p_n}{(2\pi)^d} e^{-ip_1x_1}\cdots e^{-ip_nx_n}\Big[(2\pi)^4\delta(p_1+\cdots+p_n)\tilde{G}^{(n)}_{f_1\dots f_n}(p_1,\dots,p_n)\Big], \end{aligned} \tag{6.2}$$

where we denoted the (bosonic or fermionic) i-th field by f_i and where we explicitly included the (usually omitted) total conservation of momentum delta function factor. In this context

it becomes important because (3.60) tells us that it influences mass dimensions. Indeed, using the (3.60) results, we find the mass dimensions of the previous two lines to be, respectively,

$$[G^{(n)}_{f_1\dots f_n}] = n_B(1-\varepsilon) + n_F\left(\frac{3}{2}-\varepsilon\right) = n_B + \frac{3}{2}n_F - n\varepsilon, \tag{6.3}$$

$$[G^{(n)}_{f_1\dots f_n}] = nd - d + [\tilde{G}^{(n)}_{f_1\dots f_n}], \tag{6.4}$$

where in the second line the first term comes from the n loop momenta and the second one from the delta function. Equating these two relations gives us a closed form for $[\tilde{G}^{(n)}_{f_1\dots f_n}]$, however in this thesis we are interested in 1PI n-point functions, therefore we would like to determine $[\tilde{\Gamma}^{(n)}_{f_1\dots f_n}]$ instead. This is done by observing that, for $n > 2$,

$$[\tilde{G}^{(n)}_{f_1\dots f_n}] = [\tilde{G}^{(2)}_{f_1}\cdots\tilde{G}^{(2)}_{f_n}\tilde{\Gamma}^{(n)}_{f_1\dots f_n}]. \tag{6.5}$$

To see why the above is true (the $n = 2$ case is immediate), observe the first term on the RHS of figure 2.2 to deduce that the amputated 1PI n-point bubble $\Gamma^{(n)}$ connected to n full propagators must be contained in $G^{(n)}$. All the other possibilities, such as the second RHS term of figure 2.2, must have the same mass dimension as the first one because they are summed with it. This shows that the above holds, and therefore one can compute

$$\begin{aligned}[\tilde{\Gamma}^{(n)}_{f_1\dots f_n}] &= [\tilde{G}^{(n)}_{f_1\dots f_n}] - \left([\tilde{G}^{(2)}_{f_1}] + \cdots + [\tilde{G}^{(2)}_{f_n}]\right) = [\tilde{G}^{(n)}_{f_1\dots f_n}] - (-2n_B - n_F) \\ &= \left(4 - 4n + 3n_B + \frac{5}{2}n_F\right) + (n-2)\varepsilon.\end{aligned} \tag{6.6}$$

Given a quantity q, we can partition its mass dimension like $[q] = [q]_0 + [q]_\varepsilon$, where we define $[q]_0 := [q]\big|_{\varepsilon=0}$ and $[q]_\varepsilon := [q] - [q]_0$. In the present case we have

$$\begin{cases}[\tilde{\Gamma}^{(n)}_{f_1\dots f_n}]_0 = 4 - 4n + 3n_B + \frac{5}{2}n_F \\ [\tilde{\Gamma}^{(n)}_{f_1\dots f_n}]_\varepsilon = (n-2)\varepsilon\end{cases} \tag{6.7}$$

and we see the particularly nice result of $[\tilde{\Gamma}^{(n)}_{f_1\dots f_n}]_\varepsilon$ being only dependent on n, irregardless of n_B and n_F. This last quantity is of capital importance for what immediately follows. Indeed, consider the mass dimension of a Feynman diagram computed with the algorithm described in the previous Chapter 5. Its analytical value will be proportional to the product of the coupling constants of its vertices times a linear sum of master integrals. Specifically, we saw in Chapter 5 that the 1 loop and 2 loops master integrals were given by (3.16), its square and by (3.34). What all these master integrals have in common is that their mass dimensions ε-components are uniquely determined by their power of the N constant, namely $[T^D_1]_\varepsilon = [N]$, $[(T^D_1)^2]_\varepsilon = [N^2]$ and $[I^D_{111}]_\varepsilon = [N^2]$, meaning that the mass dimension ε-component of a L-loops diagram is given by

$$[C_1^{\mu_1}\cdots C_n^{\mu_n}N^L]_\varepsilon = \sum_{i=1}^{n}\mu_i[C_i]_\varepsilon - 2L\varepsilon, \tag{6.8}$$

where $\mu_i \in \mathbb{N}$ is the amount of times the coupling constant C_i appears in the diagram. Finally, because the diagram is summed inside $\tilde{\Gamma}^{(n)}_{f_1\dots f_n}$, equating this with $[\tilde{\Gamma}^{(n)}_{f_1\dots f_n}]_\varepsilon$ gives

L	n	$[\tilde{\Gamma}^{(n)}]_0$	$[\tilde{\Gamma}^{(n)}]_\varepsilon$	ω	Combinations
0	2	2	0	0	1
0	4	0	2ε	2ε	C_4'
0	6	-2	4ε	4ε	$C_4'^2, C_6'$
1	2	2	0	2ε	C_4'
1	4	0	2ε	4ε	$C_4'^2, C_6'$
1	6	-2	4ε	6ε	$C_4'^3, C_4'C_6'$
2	2	2	0	4ε	$C_4'^2, C_6'$
2	4	0	2ε	6ε	$C_4'^3, C_4'C_6'$
2	6	-2	4ε	8ε	$C_4'^4, C_4'^2C_6', C_6'^2$

Table 6.1: Mass dimension components and authorized combinations of the C_4 and C_6 coupling constants for $n = n_B$ interacting scalar fields.

$$\sum_{i=1}^{n} \mu_i [C_i]_\varepsilon = (n - 2 + 2L)\varepsilon =: \omega(L, n), \tag{6.9}$$

where we the RHS quantity $\omega(L, n)$ constrains which and how many coupling constants can enter the LHS sum for a 1PI n-point function at L loops. The formula can also be read backwards and, given a contribution proportional to a specific product of coupling constants, tell us at which loop order L it contributes in the 1PI n-point function. Table 6.1 applies the above formula to the theory considered in this Chapter. Moreover, we now understand the reason why (3.20) and (3.35) were expressed with the modified $[\bar{N}] = 0$ constant: that way, after pulling out all the coupling constants ε-components $C_i = \mu^{[C_i]} C_i'$, these can be easily redistributed among the master integrals, and only a $[\tilde{\Gamma}^{(n)}_{f_1 \dots f_n}]_\varepsilon$ component eventually survives.

6.2 Momentum dependencies

The second point to be discussed before giving the list of results is the way we handle momentum-dependent terms. For reasons that will become clear in Chapter 7, we'd like to cast all momentum dependencies into the sum of the squares of all the external momenta. For example, in 4-point functions, we'd need to apply the identity

$$\sum_{i=2}^{4} p_{1i}^2 = 3p_1^2 + \sum_{i=2}^{4} p_i^2 + 2p_1 \sum_{i=2}^{4} p_i = \sum_{i=1}^{n} p_i^2, \tag{6.10}$$

where in the second equality conservation of momentum was used. It is specifically this latter fact that will be used in our computer-algebra program. Indeed, for a 1PI n-point function, consider all the possible momentum-dependent terms $p_1^2, (p_1 \cdot p_2), \dots, (p_1 \cdot p_n), (p_2 \cdot p_1), p_2^2, \dots, p_n^2$. The list is finite because of (4.10). By conservation of momentum $p_n = -(p_1 + \dots + p_{n-1})$, we can lift all p_n-dependencies. Next, we observe that if we accordingly square the conservation of momentum constraint we obtain

$$p_n^2 = (p_1 + p_2 + q)^2 = p_1^2 + 2(p_1 \cdot p_2) + p_2^2 + q^2 + 2q \cdot (p_1 + p_2), \tag{6.11}$$

where $q = p_3 + \cdots + p_{n-1}$, meaning

$$(p_1 \cdot p_2) = \left[p_n^2 - p_1^2 - p_2^2 - q^2 - 2q \cdot (p_1 + p_2)\right]/2. \tag{6.12}$$

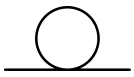
This equation effectively lifts all $(p_i \cdot p_j)$ momentum-dependencies, for $i \neq j$, because $(p_1 \cdot p_2)$ terms are obviously eliminated, the fourth term on the RHS contains all the $(p_i \cdot p_j)$ dependencies for $2 < i, j < n$, and the fifth term on the RHS contains all the $(p_1 \cdot p_i)$ and $(p_2 \cdot p_i)$ dependencies for $2 < i < n$. Finally, the first term of the RHS will reintroduce p_n^2 terms back, allowing us to group together all momentum dependencies in the desired form.

6.3 1PI 2-point diagrams

At all loops, all columns of table 6.1 are verified, in particular $[\tilde{\Gamma}^{(n)}]_0 = 2$ and $[\tilde{\Gamma}^{(n)}]_\varepsilon = 0$. The topological diagrams are equivalent to the regular ones, so no $\mathcal{P}$ is needed.

6.3.1 Results at 1 loop

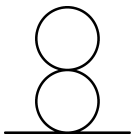
– Tadpole diagram [21, p.113]



$$= \frac{(-iC_4)}{2} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{i}{k_1^2 - m^2} = \frac{C_4' \mu^{2\varepsilon}}{2} T_1^D = iC_4' \bar{N} \frac{m^2}{2} \left[\frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right] \tag{6.13}$$

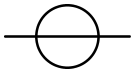
6.3.2 Results at 2 loops

– Double tadpole diagram [21, p.114]



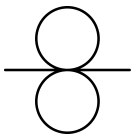
$$\begin{aligned} &= \frac{(-iC_4)^2}{4} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{i^2}{(k_1^2 - m^2)^2} \frac{i}{k_2^2 - m^2} = \frac{iC_4'^2 \mu^{4\varepsilon}}{4} I_{012}^D \\ &= \frac{iC_4'^2 \mu^{4\varepsilon}}{4} \frac{D-2}{2m^2} I_{011}^D = -iC_4'^2 \bar{N}^2 \frac{m^2}{4} \left[\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + \mathcal{O}\big(\varepsilon^0\big)\right] \end{aligned} \tag{6.14}$$

– Sunset (or London transport) diagram [21, p.117]



$$\begin{aligned} &= \frac{(-iC_4)^2}{3!} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{i}{(k_1 + p_1)^2 - m^2} \frac{i}{(k_1 - k_2)^2 - m^2} \frac{i}{k_2^2 - m^2} \\ &= \frac{iC_4'^2 \mu^{4\varepsilon}}{3!} \left(\frac{i}{2^D \pi^{D/2}}\right)^2 \big(J_{111}^D - p_1^2 J_{112}^D + 4 p_{1,\mu} p_{1,\nu} J_{113}^D [k_1^\mu k_1^\nu]\big) + \mathcal{O}\big((p_1^2)^{3/2}\big) \\ &= \frac{iC_4'^2 \mu^{4\varepsilon}}{3!} \big(I_{111}^D - p_1^2 I_{112}^D + 4 p_{1,\mu} p_{1,\nu} g^{\mu\nu} I_{123}^{D+2}\big) + \mathcal{O}\big((p_1^2)^{3/2}\big) \\ &= \frac{iC_4'^2 \mu^{4\varepsilon}}{3!} \left\{I_{111}^D + p_1^2 \left[\frac{(D-2)^2}{3Dm^4} I_{011}^D - \left(1 + \frac{4}{D}\right) \frac{D-3}{9m^2} I_{111}^D\right]\right\} + \mathcal{O}\big((p_1^2)^{3/2}\big) \\ &= iC_4'^2 \bar{N}^2 \left[-\frac{m^2}{4} \left(\frac{1}{\varepsilon^2} + \frac{3}{\varepsilon}\right) + \frac{p_1^2}{24\varepsilon} + \mathcal{O}\big(\varepsilon^0\big)\right] + \mathcal{O}\big((p_1^2)^{3/2}\big) \end{aligned} \tag{6.15}$$

– Symmetric double tadpole [12, p.51]



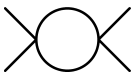
$$= \frac{-iC_6}{8\Lambda^2}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\frac{i}{k_1^2-m^2}\frac{i}{k_2^2-m^2} = \frac{iC_6}{8\Lambda^2} I^D_{011}$$
$$= -iC_6'\bar{N}^2 m^2\left(\frac{m^2}{\Lambda^2}\right)\left[\frac{1}{8\varepsilon^2}+\frac{1}{4\varepsilon}+\mathcal{O}\big(\varepsilon^0\big)\right] \tag{6.16}$$

6.4 1PI 4-point diagrams

At all loops, all columns of table 6.1 are verified, in particular $[\tilde{\Gamma}^{(n)}]_0 = 0$ and $[\tilde{\Gamma}^{(n)}]_\varepsilon = 2\varepsilon$. We write in parentheses the number of summed diagrams in the given topological one.

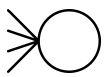
6.4.1 Results at 1 loop

– Bow tie diagram (sum of 3 diagrams) [12, p.29]



$$= \frac{(-iC_4)^2}{2}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{i}{(k_1+p_{12})^2-m^2}\frac{i}{k_1^2-m^2} + \mathcal{P}$$
$$= \frac{iC_4'^2\mu^{4\varepsilon}}{2}\big(T_2^D - p^2 T_3^D + 4p_\mu p_\nu T_4^D[k^\mu k^\nu]\big) + \mathcal{O}\big((p^2)^{3/2}\big) + \mathcal{P}$$
$$= \frac{iC_4'^2\mu^{4\varepsilon}}{2}\big(T_2^D - p^2 T_3^D - 2p_\mu p_\nu g^{\mu\nu} T_4^{D+2}\big) + \mathcal{O}\big((p^2)^{3/2}\big) + \mathcal{P}$$
$$= \frac{iC_4'^2\mu^{4\varepsilon}}{2}\left(\frac{D-2}{2m^2} - p^2\frac{(D-4)(D-2)}{24m^4}\right)T_1^D + \mathcal{O}\big((p^2)^{3/2}\big) + \mathcal{P}$$
$$= \mu^{2\varepsilon}(iC_4'^2\bar{N})\left[\frac{1}{2\varepsilon}+\frac{p^2}{12m^2}+\mathcal{O}(\varepsilon)\right] + \mathcal{O}\big((p^2)^{3/2}\big) + \mathcal{P}$$
$$= \mu^{2\varepsilon}(iC_4'^2\bar{N})\left[\frac{3}{2\varepsilon}+\frac{1}{12m^2}\sum_{i=2}^{4} p_{1i}^2+\mathcal{O}(\varepsilon)\right] + \mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}}, \mathcal{P}\Big)$$
$$= \mu^{2\varepsilon}(iC_4'^2\bar{N})\left[\frac{3}{2\varepsilon}+\frac{1}{12m^2}\sum_{i} p_i^2+\mathcal{O}(\varepsilon)\right] + \mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}}, \mathcal{P}\Big) \tag{6.17}$$

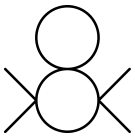
– Squid diagram [12, p.29]



$$= \frac{-iC_6}{2\Lambda^2}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{i}{k_1^2-m^2} = \frac{C_6'\mu^{4\varepsilon}}{2\Lambda^2}T_1^D = \mu^{2\varepsilon}(iC_6'\bar{N})\left(\frac{m^2}{\Lambda^2}\right)\left[\frac{1}{2\varepsilon}+\frac{1}{2}+\mathcal{O}(\varepsilon)\right] \tag{6.18}$$

6.4.2 Results at 2 loops

– Snowman diagram (sum of 3 diagrams) [12, p.52]

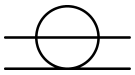


$$
\begin{aligned}
&= \frac{(-iC_4)^3}{4}\int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\frac{i}{k_1^2-m^2}\frac{i^2}{(k_2^2-m^2)^2}\frac{i}{(p_{12}-k_2)^2-m^2}+\mathcal{P}\\
&= \frac{iC_4'^3\mu^{6\varepsilon}}{4}\left(\frac{i}{2^D\pi^{D/2}}\right)^2\left(J^D_{013}-p^2J^D_{014}+4p_\mu p_\nu J^D_{015}[k_2^\mu k_2^\nu]\right)+\mathcal{O}\big((p^2)^{3/2}\big)+\mathcal{P}\\
&= \frac{iC_4'^3\mu^{6\varepsilon}}{4}\left(I^D_{013}-p^2I^D_{014}+2p_\mu p_\nu g^{\mu\nu}I^{D+2}_{025}\right)+\mathcal{O}\big((p^2)^{3/2}\big)+\mathcal{P}\\
&= \frac{iC_4'^3\mu^{6\varepsilon}}{4}\left(\frac{1}{4m^4}-p^2\frac{D-6}{48m^6}\right)(D-4)(D-2)I^D_{011}+\mathcal{O}\big((p^2)^{3/2}\big)+\mathcal{P}\\
&= \mu^{2\varepsilon}(iC_4'^3\bar{N}^2)\left[\frac{1}{4\varepsilon}+\frac{p^2}{24\varepsilon m^2}+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\big((p^2)^{3/2}\big)+\mathcal{P}\\
&= \mu^{2\varepsilon}(iC_4'^3\bar{N}^2)\left[\frac{3}{4\varepsilon}+\frac{1}{24\varepsilon m^2}\sum_{i=2}^{4}p_{1i}^2+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}},\mathcal{P}\Big)\\
&= \mu^{2\varepsilon}(iC_4'^3\bar{N}^2)\left[\frac{3}{4\varepsilon}+\frac{1}{24\varepsilon m^2}\sum_{i}p_i^2+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}},\mathcal{P}\Big)
\end{aligned}
\tag{6.19}
$$

– Double bow tie diagram [12, p.52]

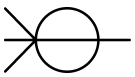
$$
= \mu^{2\varepsilon}(iC_4'^3\bar{N}^2)\left[-\frac{3}{4\varepsilon^2}-\frac{1}{12\varepsilon m^2}\sum_i p_i^2+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}},\mathcal{P}\Big) \tag{6.20}
$$

– Extended sunset diagram [12, p.52]



$$
= \mu^{2\varepsilon}(iC_4'^3\bar{N}^2)\left[-\frac{3}{2\varepsilon^2}-\frac{3}{2\varepsilon}-\frac{1}{6\varepsilon m^2}\sum_i p_i^2+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}},\mathcal{P}\Big) \tag{6.21}
$$

– Dart diagram (sum of 4 diagrams) [12, p.53]

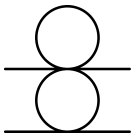


$$
= \mu^{2\varepsilon}(iC_4'C_6'\bar{N}^2)\left[-\frac{m^2}{\Lambda^2}\left(\frac{1}{\varepsilon^2}+\frac{3}{\varepsilon}\right)+\frac{1}{24\varepsilon\Lambda^2}\sum_i p_i^2+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{123}^2)^{\frac{3}{2}},\mathcal{P}\Big) \tag{6.22}
$$

– Double squid diagram [12, p.53]

$$
= -\mu^{2\varepsilon}(iC_4'C_6'\bar{N}^2)\left(\frac{m^2}{\Lambda^2}\right)\left[\frac{1}{4\varepsilon^2}+\frac{1}{4\varepsilon}+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{1234}^2)^{\frac{3}{2}},\mathcal{P}\Big) \tag{6.23}
$$

– Four-armed snowman diagram (sum of 6 diagrams) [12, p.53]



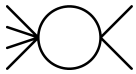
$$
= -\mu^{2\varepsilon}(iC_4'C_6'\bar{N}^2)\left[\frac{3}{2}\frac{m^2}{\Lambda^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\right)+\frac{1}{12\varepsilon\Lambda^2}\sum_i p_i^2+\mathcal{O}\big(\varepsilon^0\big)\right]+\mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}},\mathcal{P}\Big) \tag{6.24}
$$

6.5 1PI 6-point diagrams

At all loops, all columns of table 6.1 are verified, in particular $[\tilde{\Gamma}^{(n)}]_0 = -2$ and $[\tilde{\Gamma}^{(n)}]_\varepsilon = 4\varepsilon$. Also here, we write in parentheses the number of diagrams that are being summed in the given topological one.

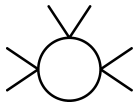
6.5.1 Results at 1 loop

– One C_4 and one C_6 interactions diagram (sum of 15 diagrams) [12, p.32]



$$= \mu^{4\varepsilon} \frac{iC_4' C_6' \bar{N}}{\Lambda^2} \left[\frac{15}{2\varepsilon} + \frac{1}{3m^2} \sum_i p_i^2 + \mathcal{O}(\varepsilon) \right] + \mathcal{O}\Big((p_{1234}^2)^{\frac{3}{2}}, \mathcal{P} \Big) \tag{6.25}$$

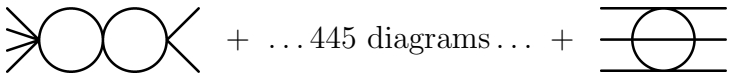
– Three C_4 interactions diagram (sum of 15 diagrams)



$$= \mu^{4\varepsilon} \frac{iC_4'^3 \bar{N}}{m^2} \left[-\frac{15}{2} - \frac{1}{2m^2} \sum_i p_i^2 + \mathcal{O}(\varepsilon) \right] + \mathcal{O}\Big((p_{1234}^2)^{\frac{3}{2}}, \mathcal{P} \Big) \tag{6.26}$$

6.5.2 Results at 2 loops

– All 2 loops 6-point diagrams (sum of 445 diagrams)



$$\begin{aligned} &= \mu^{4\varepsilon} i \bar{N}^2 \left[\frac{C_4'^4}{m^2} \left(\frac{30}{\varepsilon} + \frac{7}{4\varepsilon m^2} \sum_i p_i^2 \right) - \frac{C_4'^2 C_6'}{\Lambda^2} \left(\frac{135}{4\varepsilon^2} + \frac{15}{4\varepsilon} + \frac{25}{12\varepsilon m^2} \sum_i p_i^2 \right) + \mathcal{O}(\varepsilon^0) \right] \\ &\quad + \mathcal{O}\Big((p_{12}^2)^{\frac{3}{2}}, (p_{123}^2)^{\frac{3}{2}}, (p_{1234}^2)^{\frac{3}{2}}, (p_{12345}^2)^{\frac{3}{2}}, \mathcal{P} \Big) \end{aligned} \tag{6.27}$$

Chapter 7

Renormalization of ϕ^4-theory and of the ϕ^6-EFT

In this Chapter we perform the renormalization procedure of the renormalizable ϕ^4-theory and of our first EFT, denoted by ϕ^6-EFT. For both of them, we compute all their RGEs together with all their beta functions and anomalous dimensions up to two loops contributions. For the ϕ^6-EFT, it will also be necessary to determine its Green's and physical basis.

7.1 Beta functions of ϕ^4-theory

7.1.1 Definition of the theory

As a first instructive example, we consider the renormalizable non-EFT theory given by the following Lagrangian, both in its renormalized and bare versions [8, p.116],

$$\begin{aligned}\mathcal{L} &:= \left[\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2\right] - \frac{\lambda}{4!}\phi^4 + \mathcal{L}_{\text{CT}} \\ &= \left[\frac{1}{2}(\partial\phi)^2 + \frac{\delta_\phi}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{m^2\delta_{m^2}}{2}\phi^2\right] - \frac{\lambda}{4!}\phi^4 - \frac{\lambda\delta_\lambda}{4!}\phi^4 \\ &= \left[\frac{Z_\phi}{2}(\partial\phi)^2 - \frac{m^2 Z_{m^2}}{2}\phi^2\right] - \frac{\lambda Z_\lambda}{4!}\phi^4 \\ &= \left[\frac{1}{2}(\partial\phi_B)^2 - \frac{m_B^2}{2}\phi_B^2\right] - \frac{\lambda_B}{4!}\phi_B^4,\end{aligned} \tag{7.1}$$

where in this section we denote $\lambda = C_4$ and where, using (3.44), the bare quantities are linked to the renormalized ones via [8, p.116]

$$\phi_B = \phi\sqrt{Z_\phi}, \qquad m_B^2 = m^2\frac{Z_{m^2}}{Z_\phi} = m^2\tilde{Z}_{m^2}, \qquad \lambda_B = \lambda\frac{Z_\lambda}{Z_\phi^2} = \lambda\tilde{Z}_\lambda. \tag{7.2}$$

This theory is renormalizable because it only contains 2-dimensional and 4-dimensional operators, that's why we could immediately assign a renormalization constant to all parameters.

7.1.2 Renormalization

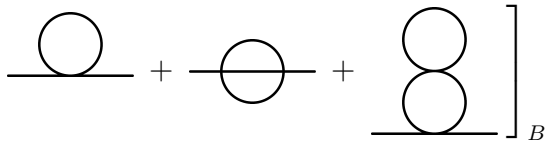
We now have to determine which 1PI n-point functions diverge. Since there's only one interaction vertex of index of divergence

$$\Delta_\lambda = \Delta\left(\times\right) = -2\varepsilon = -[\lambda], \tag{7.3}$$

then only 2-point and 4-point functions diverge, because at $d = 4$ the superficial degree of divergence (3.57) of any diagram is [8, p.107]

$$\Delta = 4 - E_\phi \geq 0 \quad \Leftrightarrow \quad E_\phi \leq 4. \tag{7.4}$$

The 1PI 2-point function is given by the diagrams [21, p.112]

$\Gamma^{(2)}_B(p) = ($ $)^{-1}_B -$  $+ \mathcal{O}(\lambda_B^3),$ (7.5)

while the 1PI 4-point function by the diagrams [21, p.112]

$\Gamma^{(4)}_B(\{p_i\}) =$  $+ \mathcal{O}(\lambda_B^4).$ (7.6)

Notice how, for the diagrams in the 1PI 2-point function, their number of loops corresponds to their number of vertices N_{λ_B}, hence to their powers of the coupling constant λ_B, while for the diagrams in the 1PI 4-point function their number of loops corresponds to $(N_{\lambda_B} - 1)$, hence the necessity to truncate at $\mathcal{O}(\lambda_B^4)$ in the latter and not at $\mathcal{O}(\lambda_B^3)$ like in the former. Another way to see it is by looking at table 6.1: at two loops, because of $[\lambda_B] = [C_{4,B}] = 2\varepsilon$, 1PI 2-point functions can contain contributions up to an $\omega(2,2) = 4\varepsilon$, corresponding to λ_B^2 hence a $\mathcal{O}(\lambda_B^3)$ truncation, while 1PI 4-point functions up to an $\omega(2,4) = 6\varepsilon$, corresponding to λ_B^3 hence a $\mathcal{O}(\lambda_B^4)$ truncation. In this section, unlike in the rest of the thesis, we decided to fuse the coefficients of the renormalization constants series with their associated powers of ε, leading to the notations

$$Z_\phi = 1 + \sum_{i=1}^{\infty} a_i \lambda'^i, \qquad Z_{m^2} = 1 + \sum_{i=1}^{\infty} b_i \lambda'^i, \qquad Z_\lambda = 1 + \sum_{i=1}^{\infty} c_i \lambda'^i, \tag{7.7}$$

that is, the ε-poles are contained inside the $\{a_i, b_i, c_i\}$ coefficients.

It is instructive to carefully carry on all the calculations by hand at least once to fully understand the logic of the renormalization procedure. Using the previously obtained results of Chapter 6, let's expand the expressions of the renormalized 1PI 2-point and 1PI 4-point functions, throwing away all finite parts hence keeping only diverging parts. Notice that by construction a_i, b_i, c_i are diverging too, hence finite parts multiplied by them must be accounted for. For the 1PI 2-point function we get

$$\begin{aligned}
\Gamma^{(2)}_R(p) &= Z_\phi \Gamma^{(2)}_B = Z_\phi \big[\Delta_B^{-1}(p) - \Sigma_B(p)\big] = -iZ_\phi\big(p^2 - \sigma_B^2\big) - Z_\phi \Bigg\{ i\lambda'_B \bar{N}_B \frac{\sigma_B^2}{2}\left[\frac{1}{\varepsilon} + 1\right] \\
&\quad - i\lambda'^2_B \bar{N}_B^2 \frac{\sigma_B^2}{4}\left[\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right] + i\lambda'^2_B \bar{N}_B^2 \left[-\frac{\sigma_B^2}{4}\left(\frac{1}{\varepsilon^2} + \frac{3}{\varepsilon}\right) + \frac{p^2}{24\varepsilon}\right]\Bigg\} + \mathcal{O}\Big(\varepsilon^0, \lambda'^3, (p^2)^{\frac{3}{2}}\Big) \\
&= -ip^2 + i\sigma^2 + i\lambda'\left[-p^2 a_1 + \sigma^2\left(b_1 - \frac{1}{2\varepsilon Q}\right)\right] \\
&\quad + i\lambda'^2\Bigg\{ -p^2\left(a_2 + \frac{1}{24\varepsilon Q^2}\right) + \sigma^2\left[a_1\left(\frac{1}{\varepsilon Q} + \frac{1}{2Q} + \frac{\ln(r)}{Q}\right) - b_1\left(\frac{1}{2\varepsilon Q} + \frac{\ln(r)}{2Q}\right)\right. \\
&\quad \left. - c_1\left(\frac{1}{2\varepsilon Q} + \frac{1}{2Q} + \frac{\ln(r)}{2Q}\right) + b_2 + \frac{1}{2\varepsilon^2 Q^2} + \frac{\ln(r)}{\varepsilon Q^2} + \frac{1}{\varepsilon Q^2}\right]\Bigg\} + \mathcal{O}\Big(\varepsilon^0, \lambda'^3, (p^2)^{\frac{3}{2}}\Big),
\end{aligned} \tag{7.8}$$

while for the 1PI 4-point function we get

$$\begin{aligned}\Gamma_R^{(4)}(\{p_i\}) = Z_\phi^2\Gamma_B^{(4)}(\{p_i\}) = \mu^{2\varepsilon}Z_\phi^2\Bigg\{ &- i\lambda'_B + (i\lambda'^2_B\bar{N}_B)\left[\frac{3}{2\varepsilon} + \frac{1}{12\sigma_B^2}\sum_i p_i^2\right] \\ &+ (i\lambda'^3_B\bar{N}_B^2)\left[-\frac{3}{4\varepsilon^2} - \frac{1}{12\sigma_B^2\varepsilon}\sum_i p_i^2\right] + (i\lambda'^3_B\bar{N}_B^2)\left[\frac{3}{4\varepsilon} + \frac{1}{24\sigma_B^2\varepsilon}\sum_i p_i^2\right] \\ &+ (i\lambda'^3_B\bar{N}_B^2)\left[-\frac{3}{2\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{1}{6\sigma_B^2\varepsilon}\sum_i p_i^2\right]\Bigg\} + \mathcal{O}\Big(\varepsilon^0, \lambda'^4, (p_{12}^2)^{\frac{3}{2}}, (p_{13}^2)^{\frac{3}{2}}, (p_{14}^2)^{\frac{3}{2}}\Big) \\ = \mu^{2\varepsilon}\Bigg\{ &- i\lambda' + i\lambda'^2\left(\frac{3}{2\varepsilon Q} - c_1\right) + i\lambda'^3\Bigg\{\frac{1}{\sigma^2}\Big(\sum_i p_i^2\Big)\Big(- \frac{a_1}{12Q} - \frac{b_1}{12Q} + \frac{c_1}{6Q} \\ &- \frac{5}{24\varepsilon Q^2}\Big) - \left[a_1\left(\frac{3}{\varepsilon Q} - \frac{3}{2Q} + \frac{3\ln(r)}{Q}\right) + \frac{3b_1}{2Q} - c_1\left(\frac{3}{\varepsilon Q} + \frac{3\ln(r)}{Q}\right) + c_2\right. \\ &\left. + \frac{9}{4\varepsilon^2Q^2} + \frac{3}{4\varepsilon Q^2} + \frac{9\ln(r)}{2\varepsilon Q^2}\right]\Bigg\}\Bigg\} + \mathcal{O}\Big(\varepsilon^0, \lambda'^4, (p_{12}^2)^{\frac{3}{2}}, (p_{13}^2)^{\frac{3}{2}}, (p_{14}^2)^{\frac{3}{2}}\Big).\end{aligned} \tag{7.9}$$

We see that at tree level there aren't any divergences. At one loop, we can deduce from $\Gamma_R^{(2)}$ the values of $a_1 = 0$ and b_1, and from $\Gamma_R^{(4)}$ the value of c_1, such that divergences cancel and the two 1PI n-point functions become finite. At two loops, $\Gamma_R^{(2)}$ provides us with the value of a_2 and we observe the presence of logarithmic poles: however, by log-cancellation, we see that when finding the value of b_2 they indeed vanish

$$\begin{aligned} b_2 &= b_1\left(\frac{1}{2\varepsilon Q} + \frac{\ln(r)}{2Q}\right) + c_1\left(\frac{1}{2\varepsilon Q} + \frac{1}{2Q} + \frac{\ln(r)}{2Q}\right) - \frac{1}{2\varepsilon^2Q^2} - \frac{\ln(r)}{\varepsilon Q^2} - \frac{1}{\varepsilon Q^2} \\ &= \left(\frac{\ln(r)}{4\varepsilon Q^2} + \frac{3\ln(r)}{4\varepsilon Q^2} - \frac{4\ln(r)}{4\varepsilon Q^2}\right) - \frac{2}{4\varepsilon^2Q^2} + \frac{1}{4\varepsilon^2Q^2} + \frac{3}{4\varepsilon^2Q^2} + \frac{3}{4\varepsilon Q^2} - \frac{4}{4\varepsilon Q^2} \\ &= \frac{1}{2\varepsilon^2Q^2} - \frac{1}{4\varepsilon Q^2}, \end{aligned} \tag{7.10}$$

as they should. By a similar calculation the logarithmic poles of $\Gamma_R^{(4)}$ cancel too, giving us the value of c_2. Having found all the $\{a_i, b_i, c_i\}$ coefficients up to two loops contributions, we can stitch everything together to obtain the renormalization constants

$$\begin{cases} Z_\phi = 1 - \dfrac{\lambda'^2}{24Q^2\varepsilon} + \mathcal{O}\big(\lambda'^3\big) \\ Z_{m^2} = 1 + \dfrac{\lambda'}{2Q\varepsilon} + \left(\dfrac{1}{2\varepsilon^2} - \dfrac{1}{4Q^2\varepsilon}\right)\lambda'^2 + \mathcal{O}\big(\lambda'^3\big) \\ Z_\lambda = 1 + \dfrac{3}{2Q\varepsilon}\lambda' + \left(\dfrac{9}{4\varepsilon^2} - \dfrac{3}{2Q^2\varepsilon}\right)\lambda'^2 + \mathcal{O}\big(\lambda'^3\big) \end{cases} \tag{7.11}$$

which agree with [13, p.146] and [21, p.148].

7.1.3 Renormalization group equations

The renormalization constants are then combined into extended renormalization constants [13, p.107]

$$\tilde{Z}_{m^2} = \frac{Z_{m^2}}{Z_\phi} = 1 + \frac{\lambda'}{2\varepsilon Q} + \left(\frac{1}{2\varepsilon^2 Q^2} - \frac{5}{24\varepsilon Q^2}\right)\lambda'^2 + \mathcal{O}(\lambda'^3), \tag{7.12}$$

$$\tilde{Z}_\lambda = \frac{Z_\lambda}{Z_\phi^2} = 1 + \frac{3}{2\varepsilon Q}\lambda' + \left(\frac{9}{4\varepsilon^2 Q^2} - \frac{17}{12\varepsilon Q^2}\right)\lambda'^2 + \mathcal{O}(\lambda'^3), \tag{7.13}$$

which should be plugged in (3.62) to obtain a system of two RGEs, whose solutions are γ_m and β_λ. Because there's no m-dependence in $\tilde{Z}_\lambda$, the system can be fully resolved by first solving the single RGE of λ, and then injecting β_λ in the RGE of m, or better still, directly use the definition of γ_m together with $\tilde{Z}_{m^2}$, the latter being only λ-dependent. In this section we compute everything explicitly, so the RGE associated to λ is given by [8, p.120]

$$\begin{aligned} 0 = \frac{\mathrm{d}\lambda_B}{\mathrm{d}\ln(\mu)} &= \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\left(\lambda'\mu^{2\varepsilon}\tilde{Z}_\lambda\right) = \frac{\mathrm{d}\lambda'}{\mathrm{d}\ln(\mu)}\mu^{2\varepsilon}\tilde{Z}_\lambda + 2\varepsilon\lambda'\mu^{2\varepsilon}\tilde{Z}_\lambda + \lambda'\mu^{2\varepsilon}\frac{\mathrm{d}\tilde{Z}_\lambda}{\mathrm{d}\lambda'}\frac{\mathrm{d}\lambda'}{\mathrm{d}\ln(\mu)} \\ &= \mu^{2\varepsilon}\Big[\beta_\lambda\tilde{Z}_\lambda + 2\varepsilon\lambda'\tilde{Z}_\lambda + \beta_\lambda\lambda'\big(\tilde{z}_1 + 2\tilde{z}_2\lambda' + \mathcal{O}(\lambda'^2)\big)\Big], \end{aligned} \tag{7.14}$$

where, similarly as before, we denoted $\tilde{Z}_\lambda = 1 + \tilde{z}_1\lambda' + \tilde{z}_2\lambda'^2 + \mathcal{O}(\lambda'^3)$.

7.1.4 Beta functions

The above equation has now to be solved. It is not difficult to see that its solution is

$$\begin{aligned} \beta_\lambda &= \frac{-2\varepsilon\lambda'\tilde{Z}_\lambda}{1 + \Big[(\tilde{Z}_\lambda - 1) + \tilde{z}_1\lambda' + 2\tilde{z}_2\lambda'^2 + \mathcal{O}(\lambda'^3)\Big]} \\ &= -2\varepsilon\lambda'\big(1 + \tilde{z}_1\lambda' + \tilde{z}_2\lambda'^2 + \mathcal{O}(\lambda'^3)\big)\big(1 - 2\tilde{z}_1\lambda' + (4\tilde{z}_1^2 - 3\tilde{z}_2)\lambda'^2 + \mathcal{O}(\lambda'^3)\big) \\ &= -2\varepsilon\lambda'\big(1 - \tilde{z}_1\lambda' + 2(\tilde{z}_1^2 - \tilde{z}_2)\lambda'^2\big) + \mathcal{O}(\lambda'^4), \end{aligned} \tag{7.15}$$

where the (2.4) expansion was used. With that at hand, finally one can compute the remaining anomalous dimensions

$$\begin{cases} \gamma_\phi = \dfrac{\lambda'^2}{12Q^2} + \mathcal{O}(\lambda'^3) \\ \gamma_m = \dfrac{\lambda'}{2Q} - \dfrac{5}{12Q^2}\lambda'^2 + \mathcal{O}(\lambda'^3) \\ \beta_\lambda = -2\varepsilon\lambda' + \dfrac{3}{Q}\lambda'^2 - \dfrac{17}{3Q^2}\lambda'^3 + \mathcal{O}(\lambda'^4) \end{cases} \tag{7.16}$$

which are compatible with reference [22]. Because the ϕ^4-theory Lagrangian of [22] is defined in a slightly different manner, to verify our results one must $m^2 \to -m^2$ and $Z_4 \to -Z_4$: the former can be done because γ_m is invariant under a dilation of the mass, while the latter is just a matter of definition. Also, keep in mind that the anomalous dimension γ'_m of [22] is defined differently, that is,

$$\gamma'_m := \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\ln\left(\frac{Z_{m^2}}{Z_\phi}\right) = -2\gamma_m. \tag{7.17}$$

All the obtained beta functions and anomalous dimensions are verified with our integrity check equation (3.70): here we explicitly verify β_λ. Again, because of the m-independence of $\tilde{Z}_\lambda$, one can effectively set the k-index to $k = i = C_4$, where we now switch to $C_4 = \lambda$ to avoid confusion with the λ-index. This gives [8, p.120]

$$0 = \tilde{z}_\lambda^{C_4}\bar{\beta}_{C_4} - 2C_4^2\frac{\partial \tilde{z}_{\lambda+1}^{C_4}}{\partial C_4} + C_4\frac{\partial \tilde{z}_\lambda^{C_4}}{\partial C_4}\bar{\beta}_{C_4}, \tag{7.18}$$

and by setting $\lambda = 0$ we explicitly see that $\bar{\beta}_{C_4}$, hence β_{C_4}, is fully determined by the total $1/\varepsilon$ pole of $\tilde{Z}_{C_4}$, that is

$$\bar{\beta}_{C_4} = 2C_4^2\frac{\partial \tilde{z}_1^{C_4}}{\partial C_4} = 2C_4^2\frac{\partial}{\partial C_4}\left[\frac{3}{2Q}C_4 - \frac{17}{12Q^2}C_4^2 + \mathcal{O}\big(C_4^3\big)\right] = \frac{3}{Q}C_4^2 - \frac{17}{3Q^2}C_4^3 + \mathcal{O}\big(C_4^4\big). \tag{7.19}$$

The more general but more cumbersome integrity check equation can also be applied. By the same argument as before one can effectively regard C_4 as the only parameter of the theory, leading to

$$0 = \sum_{l_1=0}^{\lambda_1}(l_1+1)\Big[\big(\tilde{z}_\lambda^{C_4}\big)_{l_1}\big(\bar{\beta}_{C_4}\big)_{\lambda_1-l_1}\Big] - 2(\lambda_1-1)\big(\tilde{z}_{\lambda+1}^{C_4}\big)_{\lambda_1-1}. \tag{7.20}$$

For indices of the form $(\lambda, \lambda_1) = (0, \lambda_1)$, because $(\tilde{z}_0^{C_4})_{l_1} = \delta_{l_1,0}$, this becomes

$$\big(\bar{\beta}_{C_4}\big)_{\lambda_1} = 2(\lambda_1-1)\big(\tilde{z}_1^{C_4}\big)_{\lambda_1-1}, \tag{7.21}$$

which is verified for all $\lambda_1 \leq 3$.

7.2 Beta functions of the ϕ^6-EFT

7.2.1 Definition of the EFT

Having seen the full machinery at work on the well-known ϕ^4-theory, we can apply it to our first EFT. Start with the full theory [17, p.5]

$$\mathcal{L} = \left[\frac{1}{2}(\partial\phi_S)^2 - \frac{1}{2}m_S^2\phi_S^2\right] + \left[\frac{1}{2}(\partial\phi_H)^2 - \frac{1}{2}M^2\phi_H^2\right] - \frac{\lambda_0}{4!}\phi_S^4 - \frac{\lambda_2}{4}\phi_S^2\phi_H^2 - \frac{\lambda_4}{4!}\phi_H^4 + \mathcal{L}_{\text{CT}}, \quad (7.22)$$

containing a soft field ϕ_S of mass m_S, a hard field ϕ_H of mass $M \gg m_S$, a 4-point interaction among soft modes of coupling constant λ_0, its hard modes equivalent of coupling constant λ_4 and finally an interaction vertex among soft and hard modes of coupling constant λ_2. The full theory manifestly possesses a global (discrete) symmetry of $\phi_S \to -\phi_S$ and $\phi_H \to -\phi_H$ [17, p.5]. We now wish to find the EFT of this theory by removing the hard modes: to do so we consider the OPE [17, p.8]

$$\mathcal{L}_{\phi^6\text{-EFT}} := \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{C_4}{4!}\phi^4 + \sum_{[O_j]>d} \frac{C_j}{\Lambda^{[O_j]-d}} O_j + \mathcal{L}_{\text{CT}}, \quad (7.23)$$

where $\phi \neq \phi_S$, $m \neq m_S$ and $C_4 \neq \lambda_0$ are unrelated. We'll call it the ϕ^6-EFT.

7.2.2 Obtaining the Green's basis

Now, we must determine a Green's basis for the above OPE. Because of Lorentz invariance, only an even number of derivatives $d_j \in 2\mathbb{N}$ can appear inside the O_j operators [17, p.8]. Therefore, since we consider operators up to dimension 6, it can only be $d_j \in \{0, 2, 4\}$. The ϕ^6-EFT must also inherit the $\phi \to -\phi$ global symmetry of its original full theory, meaning that only an even number of ϕ fields can appear in the OPE [17, p.8]. This ultimately rules out any 5-dimensional operators in the OPE, and as a result only three classes of operators can appear in it: 6 fields with 0 derivatives, 4 fields with 2 derivatives, or 2 fields with 4 derivatives. In the following, we drop all derivative indices $\partial \equiv \partial^\mu \equiv \partial_\mu \equiv \partial^\nu$ because, as they are dummy indices, one can always choose them accordingly.

Operators made of 6 fields and 0 derivatives

Trivially ϕ^6 is the only possible operator, which is therefore an element of the Green's basis [17, p.9].

Operators made of 4 fields and 2 derivatives

There are two sub-classes of operators: the class where the two derivatives act one after the other on a single object, and the class where they act independently on two different objects. The elements of the first class are of the form

$$\partial(\phi^a \partial\phi^b)\phi^c, \quad (7.24)$$

provided $a, c \in \mathbb{N}$, $b \in \mathbb{N}^*$ and $a + b + c = 4$, while the elements of the second class are

$$(\partial\phi^a)(\partial\phi^b)\phi^c, \quad (7.25)$$

a	b	c	Operator	Equivalence
0	4	0	$\Box\phi^4$	0
1	3	0	$\partial(\phi\partial\phi^3)$	0
2	2	0	$\partial(\phi^2\partial\phi^2)$	0
3	1	0	$\partial(\phi^3\partial\phi)$	0
0	3	1	$\phi\Box\phi^3$	$\phi\Box\phi^3$
1	2	1	$\partial(\phi\partial\phi^2)\phi$	$\phi^2(\partial\phi)^2+\phi^2\Box\phi^2$
2	1	1	$\partial(\phi^2\partial\phi)\phi$	$\phi^2(\partial\phi)^2+\phi^3\Box\phi$
0	2	2	$\phi^2\Box\phi^2$	$\phi^2\Box\phi^2$
1	1	2	$\partial(\phi\partial\phi)\phi^2$	$\phi^2(\partial\phi)^2+\phi^3\Box\phi$
0	1	3	$\phi^3\Box\phi$	$\phi^3\Box\phi$
1	3	0	$(\partial\phi)(\partial\phi^3)$	$(\partial\phi)(\partial\phi^3)$
2	2	0	$(\partial\phi^2)(\partial\phi^2)$	$(\partial\phi^2)(\partial\phi^2)$
1	2	1	$(\partial\phi)(\partial\phi^2)\phi$	$\phi^2(\partial\phi)^2$
1	1	2	$(\partial\phi)(\partial\phi)\phi^2$	$\phi^2(\partial\phi)^2$

Table 7.1: All possible operators of dimension 6 composed of 4 fields ϕ and 2 derivatives. The horizontal line separates the two sub-classes. In the last column, the + sign signifies a linear combination of its two terms.

where this time $a,b\in\mathbb{N}^*$, $c\in\mathbb{N}$ and again $a+b+c=4$. Notice that, for the former, setting $c=0$ gives total derivatives and setting $a=0$ gives operators with the box operator, while for the latter there's a symmetry $a\leftrightarrow b$. Table 7.1 groups all the possible operators of both sub-classes respecting their respective constraints on a,b,c. Because the first 4 entries are total derivatives, we know that they all are equivalent to the null operator: using this fact, we can derive the identities

$$3\phi^2(\partial\phi)^2\equiv-\phi^3\Box\phi, \tag{7.26}$$
$$(\partial\phi)(\partial\phi^3)\equiv-\phi\Box\phi^3, \tag{7.27}$$
$$\phi^2\Box\phi^2\equiv-2\phi(\partial\phi)(\partial\phi^2)=-4\phi^2(\partial\phi)^2=-(\partial\phi^2)(\partial\phi^2), \tag{7.28}$$
$$\phi^3\Box\phi\equiv-3\phi^2(\partial\phi)^2=-(\partial\phi)(\partial\phi^3), \tag{7.29}$$

which, accordingly chained one after the other, show that

$$\phi^3\Box\phi\equiv-3\phi^2(\partial\phi)^2\equiv-(\partial\phi)(\partial\phi^3)\equiv\phi\Box\phi^3\equiv\frac{3}{4}\phi^2\Box\phi^2\equiv-\frac{3}{4}(\partial\phi^2)(\partial\phi^2). \tag{7.30}$$

The last column of table 7.1 carries on the chain rule on all the other operators and it therefore shows the linear combinations of operators that are equivalent to the original ones. Clearly any operator of table 7.1 is either null or is a linear combination of (7.30) operators, which are all equivalent among themselves, therefore all operators of this class are equivalent to one of (7.30) [17, p.9]. We'll arbitrarily pick $\phi^3\Box\phi$ to be an element of the Green's basis.

Operators made of 2 fields and 4 derivatives

In order to find all operators of this class, we devise a simple algorithm that automatically generates them all. We consider an unordered list of two objects, and at each step of the algorithm we have the choice to either multiply their content to get a single object or to give singularly to each object of the list a derivative. The algorithm ends when all 4 derivatives are given: if at that stage the list is still composed of two objects, multiply them as a final step. So, in our case, the algorithm would proceed as

$$\{(\phi,\phi)\} \;\to\; \{\Box^2\phi^2,(\partial\phi,\phi)\} \;\to\; \{\Box^2\phi^2,\partial^3[(\partial\phi)\phi],(\Box\phi,\phi),(\partial\phi,\partial\phi)\} \;\to\; \dots \tag{7.31}$$

and, after a few iterations, it would end and give us the set of all possible operators

$$\begin{gathered}\phi\Box^2\phi, \quad (\Box\phi)^2, \quad (\partial\phi)(\partial^3\phi),\\ \partial\big(\phi\partial^3\phi\big), \quad \partial[(\partial\phi)\Box\phi], \quad \Box\big[(\partial\phi)^2\big], \quad \Box(\phi\Box\phi), \quad \partial^3(\phi\partial\phi), \quad \Box^2\phi^2,\end{gathered} \tag{7.32}$$

where we put all total derivatives on the second line. We therefore only need to show that the first three operators on the first line are equivalent: this is easily done by carrying out the chain rule on the first two operators of the second line, showing us that indeed [17, p.9]

$$(\Box\phi)^2 \equiv -(\partial\phi)(\partial^3\phi) \equiv \phi\Box^2\phi. \tag{7.33}$$

Again, we'll arbitrarily pick $(\Box\phi)^2$ to be an element of the Green's basis.

Obtained Green's basis

In conclusion, the Green's basis for the OPE of the ϕ^6-EFT up to dimension 6 is the minimal non-redundant set of operators [17, p.9]

$$G^{(6)}_{\phi^6\text{-EFT}} = \Big\{(\Box\phi)^2,\, \phi^3\Box\phi,\, \phi^6\Big\}, \tag{7.34}$$

sometimes referred to as in the **box form** [1, p.17], resulting in the EFT Lagrangian

$$\begin{aligned}\mathcal{L}_{\phi^6\text{-EFT}} = {}& \frac{Z_\phi}{2}(\partial\phi)^2 - \frac{m^2 Z_{m^2}}{2}\phi^2 - \frac{C_4 Z_{C_4}}{4!}\phi^4\\ &+ \sum_{O_j\in P^{(6)}_{\phi^6\text{-EFT}}} \frac{Z_{C_j}C_j}{\Lambda^2}O_j + \sum_{O_j\in\left(G^{(6)}_{\phi^6\text{-EFT}}\setminus P^{(6)}_{\phi^6\text{-EFT}}\right)} \frac{\hat{\delta}_j}{\Lambda^2}O_j.\end{aligned} \tag{7.35}$$

7.2.3 Obtaining the physical basis

The above Lagrangian requires us to find the physical basis of the ϕ^6-EFT. A field redefinition of the form (4.11) affects the EFT Lagrangian like

$$\begin{aligned}\mathcal{L}_{\phi^6\text{-EFT}} &\to \frac{Z_\phi}{2}\left(\partial\phi+\frac{\partial f}{\Lambda^2}\right)^2 - \frac{m^2Z_{m^2}}{2}\left(\phi+\frac{f}{\Lambda^2}\right)^2 - \frac{C_4Z_{C_4}}{4!}\left(\phi+\frac{f}{\Lambda^2}\right)^4 + (\text{OPE})\\ &= \frac{Z_\phi}{2}\left((\partial\phi)^2+\frac{2(\partial\phi)f}{\Lambda^2}\right) - \frac{m^2Z_{m^2}}{2}\left(\phi^2+\frac{2f}{\Lambda^2}\phi\right) - \frac{C_4Z_{C_4}}{4!}\left(\phi^4+\frac{4f}{\Lambda^2}\phi^3\right) + (\text{OPE})\\ &= \mathcal{L}_{\phi^6\text{-EFT}} + Z_\phi(\partial\phi)\frac{\partial f}{\Lambda^2} - \left(m^2Z_{m^2}\phi + \frac{C_4Z_{C_4}}{3!}\phi^3\right)\frac{f}{\Lambda^2},\end{aligned} \tag{7.36}$$

where no term in the OPE gets an additional contribution from this field redefinition because they all already carry a Λ^{-2} factor. Notice that the shift induced by the transformation is proportional to the EOMs up to $\mathcal{O}(\Lambda^{-3})$. It is clear that the only possible field redefinitions that respect both the power counting and the global symmetry are

$$f \in \Omega_{\phi^6\text{-EFT}} = \{\Box\phi, \phi^3\}, \tag{7.37}$$

therefore by (4.21) the order of the physical basis is exactly one. Indeed, as we'll see below, the $f = \Box\phi$ transformation gets rid of the $(\Box\phi)^2$ operator and the $f = \phi^3$ transformation gets rid of the $\phi^3\Box\phi$ operator. Thus, the physical basis is given by

$$P^{(6)}_{\phi^6\text{-EFT}} = \left\{\phi^6\right\} \tag{7.38}$$

and the physical Lagrangian is of the form [17, p.11]

$$\begin{aligned}\mathcal{L}_{\phi^6\text{-EFT,Phys}} &:= \frac{Z_\phi}{2}(\partial\phi)^2 - \frac{m^2 Z_{m^2}}{2}\phi^2 - \frac{C_4 Z_{C_4}}{4!}\phi^4 - \frac{C_6 Z_{C_6}}{6!\Lambda^2}\phi^6 \\ &= \frac{1}{2}(\partial\phi_B)^2 - \frac{m^2 Z_{m^2}}{Z_\phi}\frac{\phi_B^2}{2} - \frac{C_4 Z_{C_4}}{Z_\phi^2}\frac{\phi_B^4}{4!} - \frac{C_6 Z_{C_6}}{\Lambda^2 Z_\phi^3}\frac{\phi_B^6}{6!}.\end{aligned} \tag{7.39}$$

As we know from Chapter 4, the elements of $G^{(6)}_{\phi^6\text{-EFT}} \setminus P^{(6)}_{\phi^6\text{-EFT}}$ in $\mathcal{L}_{\phi^6\text{-EFT}}$ do not require a renormalization constant and must be looked at as merely improper counterterms instead, implying that the EFT Lagrangian takes its final form

$$\begin{aligned}\mathcal{L}_{\phi^6\text{-EFT}} &= \mathcal{L}_{\phi^6\text{-EFT,Phys}} - \frac{\tilde{\delta}_6}{4!\Lambda^2}\phi^3\Box\phi - \frac{\hat{\delta}_6}{2\Lambda^2}(\Box\phi)^2 \\ &\to \frac{Z_\phi}{2}(\partial\phi)^2 - \frac{m^2 Z_{m^2}}{2}\phi^2 - \left(C_4 Z_{C_4} - \frac{\tilde{\delta}_6 m^2 Z_{m^2}}{\Lambda^2 Z_\phi}\right)\frac{\phi^4}{4!} - \left(\frac{C_6 Z_{C_6}}{\Lambda^2} - \frac{5\tilde{\delta}_6 C_4 Z_{C_4}}{\Lambda^2 Z_\phi}\right)\frac{\phi^6}{6!} \\ &= \frac{1}{2}(\partial\phi_B)^2 - \frac{m^2 Z_{m^2}}{Z_\phi}\frac{\phi_B^2}{2} - \left(\frac{C_4 Z_{C_4}}{Z_\phi^2} - \frac{\tilde{\delta}_6 m^2 Z_{m^2}}{\Lambda^2 Z_\phi^3}\right)\frac{\phi_B^4}{4!} - \left(\frac{C_6 Z_{C_6}}{\Lambda^2 Z_\phi^3} - \frac{5\tilde{\delta}_6 C_4 Z_{C_4}}{\Lambda^2 Z_\phi^4}\right)\frac{\phi_B^6}{6!}.\end{aligned} \tag{7.40}$$

Notice that we (arbitrarily) decided to assign a 4! symmetry factor to $\tilde{\delta}_6$ instead of 3!, which would have been a more natural choice. In the above, the arrow symbolizes the two successive field redefinitions

$$f = -\frac{\hat{\delta}_6}{2Z_\phi}\Box\phi \qquad \text{and} \qquad f = -\frac{\tilde{\delta}_6}{4!Z_\phi}\phi^3. \tag{7.41}$$

For the first transformation, using the first equivalence of (7.33), we see that

$$Z_\phi(\partial\phi)\frac{\partial f}{\Lambda^2} = -\frac{\hat{\delta}_6}{2\Lambda^2}(\partial\phi)(\partial^3\phi) \equiv \frac{\hat{\delta}_6}{2\Lambda^2}(\Box\phi)^2, \tag{7.42}$$

therefore the transformation implies

$$\mathcal{L}_{\phi^6\text{-EFT}} \to \mathcal{L}_{\phi^6\text{-EFT}} + \frac{\hat{\delta}_6}{2\Lambda^2}(\Box\phi)^2 - \frac{1}{2}\left(\frac{\hat{\delta}_6 m^2 Z_{m^2}}{\Lambda^2 Z_\phi}\right)(\partial\phi)^2 + \left(\frac{2\hat{\delta}_6 C_4 Z_{C_4}}{\tilde{\delta}_6 Z_\phi}\right)\frac{\tilde{\delta}_6}{4!\Lambda^2}\phi^3\Box\phi, \tag{7.43}$$

where (4.20) was used to get the kinetic term. Observe that the above forces $\tilde{\delta}_6 \neq 0$. For the second transformation, using the first equivalence of (7.30), we see that

$$Z_\phi(\partial\phi)\frac{\partial f}{\Lambda^2} = -\frac{3\tilde{\delta}_6}{4!\Lambda^2}\phi^2(\partial\phi)^2 \equiv \frac{\tilde{\delta}_6}{4!\Lambda^2}\phi^3\Box\phi, \tag{7.44}$$

meaning

$$\mathcal{L}_{\phi^6\text{-EFT}} \to \mathcal{L}_{\phi^6\text{-EFT}} + \frac{\tilde{\delta}_6}{4!\Lambda^2}\phi^3\Box\phi + \left(\frac{\tilde{\delta}_6 m^2 Z_{m^2}}{\Lambda^2 Z_\phi}\right)\frac{\phi^4}{4!} + \left(\frac{5\tilde{\delta}_6 C_4 Z_{C_4}}{\Lambda^2 Z_\phi}\right)\frac{\phi^6}{6!}. \tag{7.45}$$

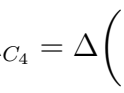
Finally, one must be aware that, even though we presented the effects of both possible field redefinitions, later in this section it will turn out that $\hat{\delta}_6 = 0$, therefore only the second field redefinition is necessary and only its action affects (7.40). That's why only $\tilde{\delta}_6$ is present in (7.40), and moreover this implies that for the renormalization procedure we only need to know its associated Feynman rule, that is

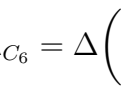
 $:= \frac{i\tilde{\delta}_6}{4\Lambda^2}\sum_{i=1}^{4} p_i^2.$ (7.46)

This shows explicitly the necessity to rewrite the momentum-dependent terms as we did in Chapter 6, so that renormalization by counterterms can be done.

7.2.4 Renormalization

To determine which 1PI n-point functions diverge, we compute the indices of divergence of the two interaction vertices

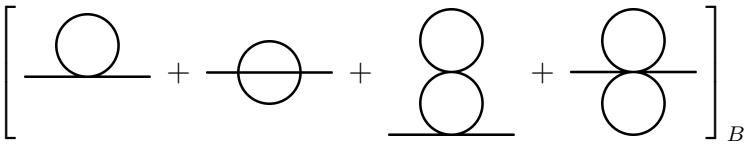
$$\Delta_{C_4} = \Delta\left(\text{  }\right) = -2\varepsilon = -[C_4], \tag{7.47}$$

$$\Delta_{C_6} = \Delta\left(\text{  }\right) = 2 - 4\varepsilon = -[C_6/\Lambda^2], \tag{7.48}$$

meaning that, for a generic diagram, at $d = 4$ we have a superficial degree of diverge

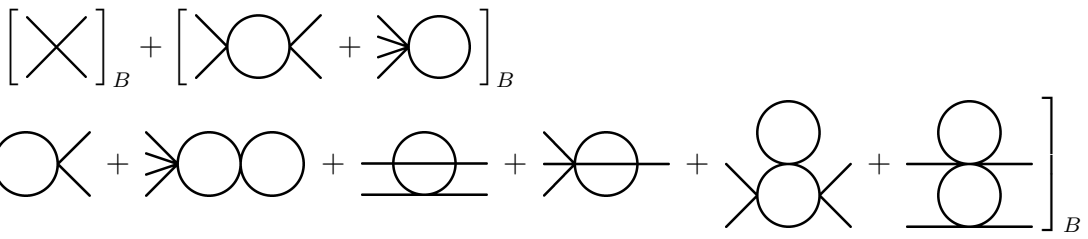
$$\Delta = 4 - E_\phi + 2N_{C_6} \geq 0 \quad \Rightarrow \quad E_\phi \leq 4 + 2N_{C_6} \leq 6 \quad \Rightarrow \quad E_\phi \leq 6, \tag{7.49}$$

being $N_{C_6} < 2$ due to the power counting formula (4.8). Indeed, with an OPE up to 6-dimensional operators, only one insertion of C_6 vertices is authorized. The 2-point 1PI function is given by the diagrams [12, p.53]

$\Gamma_B^{(2)}(p) = ($ $)_B^{-1} -$  $_B$ (7.50)

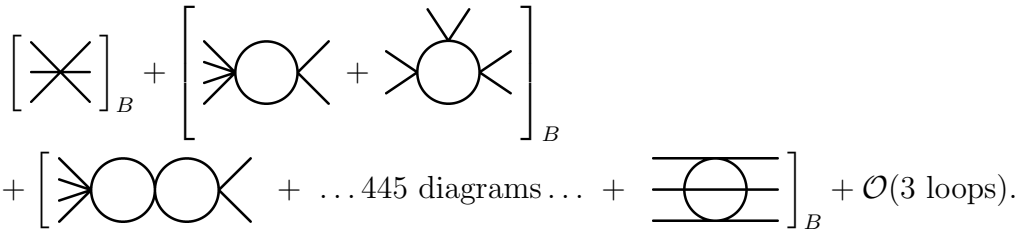
$+ \mathcal{O}(3 \text{ loops}),$

the 4-point 1PI function by the diagrams [12, p.57]

$\Gamma_B^{(4)}(\{p_i\}) =$ 

$+ \mathcal{O}(3 \text{ loops}),$ (7.51)

and the 1PI 6-point function by the diagrams

$\Gamma_B^{(6)}(\{p_i\}) =$  $+ \mathcal{O}(3 \text{ loops}).$ (7.52)

Using the results of Chapter 6, the computer-assisted renormalization of these 1PI n-point functions can be carried out. The latter is done loop order by loop order with respect to table 6.1, which unambiguously tells us which terms have to be truncated at a given loop order. After the renormalization procedure, the following renormalization constants are obtained:

$$\begin{cases} Z_\phi = 1 - \dfrac{C_4'^2}{24\varepsilon Q^2} + \mathcal{O}(3 \text{ loops}) \\ Z_{m^2} = 1 + \dfrac{C_4'}{2\varepsilon Q} + \left(\dfrac{1}{2\varepsilon^2 Q^2} - \dfrac{1}{4\varepsilon Q^2}\right) C_4'^2 + \dfrac{1}{8\varepsilon^2 Q^2}\dfrac{m^2}{\Lambda^2} C_6' + \mathcal{O}(3 \text{ loops}) \\ Z_{C_4} = 1 + \dfrac{3}{2\varepsilon Q} C_4' + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{C_6'}{C_4'} + \left(\dfrac{9}{4\varepsilon^2 Q^2} - \dfrac{3}{2\varepsilon Q^2}\right) C_4'^2 \\ \qquad + \left(\dfrac{11}{4\varepsilon^2 Q^2}\dfrac{m^2}{\Lambda^2} - \dfrac{1}{\varepsilon Q^2}\dfrac{m^2}{\Lambda^2}\right) C_6' + \mathcal{O}(3 \text{ loops}) \\ Z_{C_6} = 1 + \dfrac{15}{2\varepsilon Q} C_4' + \left(\dfrac{135}{4\varepsilon^2 Q^2} - \dfrac{75}{4\varepsilon Q^2}\right) C_4'^2 + \mathcal{O}(3 \text{ loops}) \\ \tilde{\delta}_6' = -\dfrac{C_4' C_6'}{6\varepsilon Q^2} + \mathcal{O}(3 \text{ loops}) \end{cases} \tag{7.53}$$

As a check, one can set $C_6' = 0$ to readily recover the previous renormalization constants of ϕ^4-theory. Notice the presence of a peculiar C_6'/C_4' term and the fact that all the m-dependencies are divided by an appropriate power of Λ, which renders the ratio dimensionless, as it should. We point out that, during the renormalization of $\Gamma_R^{(6)}$, it looked like some momentum-dependent divergences couldn't be cancelled and that log-cancellation didn't occur: this was because we were mistakenly considering $\mathcal{O}(\Lambda^{-4})$ terms which, obviously, must be discarded in our $\mathcal{O}(\Lambda^{-3})$ implicit prescription, so one must always be careful.

7.2.5 Renormalization group equations

Now that we have determined all the renormalization constants and all the improper counterterms, we are ready to obtain and solve the RGEs. Before doing so, as discussed in Chapter 4, we first need to transform all improper counterterms into contributions in the physical basis. We already did it in (7.40), therefore all is left to do is to substitute and take

log-derivatives (3.62) of the quantities

$$
\begin{aligned}
m^2\tilde{Z}_{m^2} &= m^2\frac{Z_{m^2}}{Z_\phi} = m^2\left[1+\frac{C_4'}{2\varepsilon Q}+\left(\frac{1}{2\varepsilon^2Q^2}-\frac{5}{24\varepsilon Q^2}\right)C_4'^2+\frac{1}{8\varepsilon^2Q^2}\frac{m^2}{\Lambda^2}C_6'+\mathcal{O}(3\text{ loops})\right],\\
C_4'\tilde{Z}_{C_4} &:= C_4'\left(\frac{Z_{C_4}}{Z_\phi^2}-\frac{\tilde{\delta}_6'm^2Z_{m^2}}{C_4'\Lambda^2Z_\phi^3}\right) = C_4'\left[1+\frac{3}{2\varepsilon Q}C_4'+\frac{1}{2\varepsilon Q}\frac{m^2}{\Lambda^2}\frac{C_6'}{C_4'}\right.\\
&\quad\left.+\left(\frac{9}{4\varepsilon^2Q^2}-\frac{17}{12\varepsilon Q^2}\right)C_4'^2+\left(\frac{11}{4\varepsilon^2Q^2}\frac{m^2}{\Lambda^2}-\frac{5}{6\varepsilon Q^2}\frac{m^2}{\Lambda^2}\right)C_6'+\mathcal{O}(3\text{ loops})\right],\\
\frac{C_6'}{\Lambda^2}\tilde{Z}_{C_6} &:= \frac{C_6'}{\Lambda^2}\left(\frac{Z_{C_6}}{Z_\phi^3}-\frac{5\tilde{\delta}_6'C_4'Z_{C_4}}{C_6'Z_\phi^4}\right)\\
&= \frac{C_6'}{\Lambda^2}\left[1+\frac{15}{2\varepsilon Q}C_4'+\left(\frac{135}{4\varepsilon^2Q^2}-\frac{427}{24\varepsilon Q^2}\right)C_4'^2+\mathcal{O}(3\text{ loops})\right],
\end{aligned}
\tag{7.54}
$$

where in the last two lines we abusively identified the resulting coefficients of the bare (7.40) operators as if they were genuine extended renormalization constants. Also, notice that in the last line we crucially grouped Λ with C_6': this is necessary for the correct application of our $\mathcal{O}(\Lambda^{-3})$ cutoff prescription because of the power counting formula (4.8), namely only one insertion of the C_6 vertex is ever allowed. Had we not done that, that is, had we removed the $1/\Lambda^2$ factors by virtue of the fact that (3.62) is equal to zero, we would have lost this information, and the truncation of the beta functions and anomalous dimensions obtained after inverting the RGEs system would have been altered, leading to divergent terms. From now on, this caveat will be implicitly conducted on all the beta functions whose parameters are associated to OPE operators.

7.2.6 Beta functions

Finally, the system of RGEs is solved and the following beta functions and anomalous dimensions are obtained:

$$
\begin{cases}
\gamma_\phi = \dfrac{C_4'^2}{12Q^2}+\mathcal{O}(3\text{ loops})\\
\beta_{m^2} = m^2\left(\dfrac{C_4'}{Q}-\dfrac{5}{6}\dfrac{C_4'^2}{Q^2}\right)+\mathcal{O}(3\text{ loops})\\
\beta_{C_4} = -2C_4'\varepsilon+3\dfrac{C_4'^2}{Q}+\dfrac{m^2}{\Lambda^2}\dfrac{C_6'}{Q}-\dfrac{17}{3}\dfrac{C_4'^3}{Q^2}-\dfrac{10}{3}\dfrac{m^2}{\Lambda^2}\dfrac{C_4'C_6'}{Q^2}+\mathcal{O}(3\text{ loops})\\
\beta_{C_6} = \dfrac{1}{\Lambda^2}\left(-4C_6'\varepsilon+15\dfrac{C_4'C_6'}{Q}-\dfrac{427}{6}\dfrac{C_4'^2C_6'}{Q^2}\right)+\mathcal{O}(3\text{ loops})
\end{cases}
\tag{7.55}
$$

where, in accordance with what was said before, the beta function of C_6 is

$$
\beta_{C_6} = \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\frac{C_6'}{\Lambda^2}
\tag{7.56}
$$

and where the anomalous dimension of the mass m can be recovered from its beta function

$$
\gamma_m = \frac{\beta_{m^2}}{2m^2} = \frac{C_4'}{2Q}-\frac{5}{12Q^2}C_4'^2+\mathcal{O}(3\text{ loops}).
\tag{7.57}
$$

Again, by setting $C_6'=0$ we can recover all the results of ϕ^4-theory. Moreover, we see that γ_ϕ and γ_m don't receive any additional contributions from the C_6 interaction vertex.

Chapter 8

Renormalization of the $\phi^6_{1,2}$-EFT

8.1 Definition of the EFT

The last theory we'll consider in this thesis is the generalization of the previous ϕ^6-EFT, but this time with two fields ϕ_1 and ϕ_2 instead of only one ϕ. Crucially, in order to be able to apply our computer-algebra program to it, ϕ_1 and ϕ_2 are fields of same masses $m_1 = m_2 = m$, meaning that the methods of Chapter 5 can be used. The newly obtained $\phi^6_{1,2}$-EFT is then defined to be

$$\mathcal{L}_{\phi^6_{1,2}\text{-EFT}} := \sum_{i=1,2}\left[\frac{1}{2}(\partial\phi_i)^2 - \frac{m_i^2}{2}\phi_i^2\right] - \frac{C_4}{4!}\phi_1^4 - \frac{D_4}{4!}\phi_2^4 - \frac{K_4}{4}\phi_1^2\phi_2^2 + \sum_{[O_j]>d}\frac{C_j}{\Lambda^{[O_j]-d}}O_j + \mathcal{L}_{\text{CT}}. \quad (8.1)$$

As before, we require the global (discrete) symmetries $\phi_i \to -\phi_i$ for both fields $i = 1, 2$. Notice that, instead, we could have asked for the symmetry $(\phi_1, \phi_2) \to -(\phi_1, \phi_2)$, where the fields would have flipped sign simultaneously, authorizing more terms in the OPE such as field-changing free propagators.

8.2 Obtaining the Green's basis

All the considerations of the previous Chapter about Lorentz invariance still apply here, therefore the classes of allowed operators in this OPE are essentially the same, with the difference that instead of a single ϕ field here we must keep track of the two distinct fields ϕ_1 and ϕ_2.

Operators made of 6 fields and 0 derivatives

It is easy to see that the only authorized operators are ϕ_1^6, $\phi_1^4\phi_2^2$, $\phi_1^2\phi_2^4$ and ϕ_2^6. Because they are not redundant among themselves, they all are elements of the Green's basis.

Operators made of 4 fields and 2 derivatives

In section 7.2.2 it was argued that there were two main sub-classes of operators, distinguished by the roles taken by the two derivatives. Obviously the same argument applies here, but with the difference that we must now fill in the integer powers of ϕ with authorized combinations of ϕ_1 and ϕ_2 fields: under this point of view, all operators of table 7.1 constitute classes over which we can build our operators for the OPE. What this means in practice is that we take table 7.1 and, operating what was described above, we generate the operators of table 8.1

and table 8.2, where the first table contains all the total derivatives and the second one the two sub-classes of operators.

Class	Operator	Full expansion
$\Box\phi^4$	$\Box(\phi_i^2\phi_j^2)$	$2\phi_i^2(\partial\phi_j)^2 + 2\phi_j^2(\partial\phi_i)^2 + 2\phi_j^2\phi_i\Box\phi_i + 2\phi_i^2\phi_j\Box\phi_j$ $+8\phi_i\phi_j(\partial\phi_i)(\partial\phi_j)$
$\partial(\phi\partial\phi^3)$	$\partial\big[\phi_i\partial(\phi_i\phi_j^2)\big]$	$2\phi_i^2(\partial\phi_j)^2 + \phi_j^2(\partial\phi_i)^2 + \phi_j^2\phi_i\Box\phi_i + 2\phi_i^2\phi_j\Box\phi_j$ $+6\phi_i\phi_j(\partial\phi_i)(\partial\phi_j)$
$\partial(\phi^2\partial\phi^2)$	$\partial\big[\phi_i^2\partial\phi_j^2\big]$	$2\phi_i^2(\partial\phi_j)^2 + 2\phi_i^2\phi_j\Box\phi_j + 4\phi_i\phi_j(\partial\phi_i)(\partial\phi_j)$
	$\partial[\phi_i\phi_j\partial(\phi_i\phi_j)]$	$\phi_i^2(\partial\phi_j)^2 + \phi_j^2(\partial\phi_i)^2 + \phi_j^2\phi_i\Box\phi_i + \phi_i^2\phi_j\Box\phi_j + 4\phi_i\phi_j(\partial\phi_i)(\partial\phi_j)$
$\partial(\phi^3\partial\phi)$	$\partial[\phi_i^2\phi_j(\partial\phi_j)]$	$\phi_i^2(\partial\phi_j)^2 + \phi_i^2\phi_j\Box\phi_j + 2\phi_i\phi_j(\partial\phi_i)(\partial\phi_j)$

Table 8.1: All possible total derivative operators of dimension 6 composed of 4 fields ϕ_i, ϕ_j and 2 derivatives. The last column gives the furthest derivative expansion through the chain rule of all operators.

Class	Operator	Equivalence
$\phi\Box\phi^3$	$\phi_i\Box(\phi_i\phi_j^2)$	$\phi_j^2\phi_i\Box\phi_i$
$\partial(\phi\partial\phi^2)\phi$	$\partial(\phi_i\partial\phi_j^2)\phi_i$	$\phi_i^2\Box\phi_j^2$
	$\partial[\phi_i\partial(\phi_i\phi_j)]\phi_j$	$\phi_i^2\Box\phi_j^2 + \phi_j^2\phi_i\Box\phi_i$
$\partial(\phi^2\partial\phi)\phi$	$\partial(\phi_i\phi_j\partial\phi_i)\phi_j$	$\phi_i^2\Box\phi_j^2$
	$\partial(\phi_i^2\partial\phi_j)\phi_j$	$\phi_i^2\Box\phi_j^2 + \phi_i^2\phi_j\Box\phi_j$
$\phi^2\Box\phi^2$	$\phi_i\phi_j\Box(\phi_i\phi_j)$	$\phi_i^2\Box\phi_j^2 + \phi_j^2\phi_i\Box\phi_i + \phi_i^2\phi_j\Box\phi_j$
	$\phi_i^2\Box\phi_j^2$	$\phi_i^2\Box\phi_j^2$
$\partial(\phi\partial\phi)\phi^2$	$\partial(\phi_i\partial\phi_i)\phi_j^2$	$\phi_j^2\Box\phi_i^2$
	$\partial(\phi_i\partial\phi_j)\phi_i\phi_j$	$\phi_i^2\Box\phi_j^2 + \phi_i^2\phi_j\Box\phi_j$
$\phi^3\Box\phi$	$\phi_i^2\phi_j\Box\phi_j$	$\phi_i^2\phi_j\Box\phi_j$
$(\partial\phi)(\partial\phi^3)$	$(\partial\phi_i)\partial(\phi_i\phi_j^2)$	$\phi_j^2\phi_i\Box\phi_i$
$(\partial\phi^2)(\partial\phi^2)$	$(\partial\phi_i^2)(\partial\phi_j^2)$	$\phi_i^2\Box\phi_j^2$
	$\partial(\phi_i\phi_j)\partial(\phi_i\phi_j)$	$\phi_i^2\Box\phi_j^2 + \phi_j^2\phi_i\Box\phi_i + \phi_i^2\phi_j\Box\phi_j$
$(\partial\phi)(\partial\phi^2)\phi$	$(\partial\phi_i)(\partial\phi_j^2)\phi_i$	$\phi_i^2\Box\phi_j^2$
	$(\partial\phi_i)\partial(\phi_i\phi_j)\phi_j$	$\phi_j^2\Box\phi_i^2 + \phi_j^2\phi_i\Box\phi_i$
$(\partial\phi)(\partial\phi)\phi^2$	$(\partial\phi_i)(\partial\phi_i)\phi_j^2$	$\phi_j^2\Box\phi_i^2 + \phi_j^2\phi_i\Box\phi_i$
	$(\partial\phi_i)(\partial\phi_j)\phi_i\phi_j$	$\phi_i^2\Box\phi_j^2$

Table 8.2: All possible non-total derivative operators of dimension 6 composed of 4 fields ϕ_i, ϕ_j and 2 derivatives. The horizontal line separates the two sub-classes. In the last column, the + sign signifies a linear combination of its terms.

Looking at table 8.1, we notice that all total derivatives, which are equivalent to the null operator, are really just different manifestations of one single identity, namely its last row

$$0 \equiv \phi_i^2(\partial\phi_j)^2 + \phi_i^2\phi_j\Box\phi_j + 2\phi_i\phi_j(\partial\phi_i)(\partial\phi_j). \tag{8.2}$$

Indeed, if we call E the above equation and $\bar{E}$ the result obtained after swapping $i \leftrightarrow j$ in E, we see that in table 8.1 the fourth expansion is given by $E + \bar{E}$, the third one by $2E$, the second one by $2E + \bar{E}$ and the first one by $2E + 2\bar{E}$. Having determined that (8.2) is the only meaningful total derivative relation, implying that it is the only one that can be used to reduce the redundant operators of table 8.2 to linear combinations of non-redundant ones, it is clear that (8.2) contains all non-redundant operators plus a redundant one. In our case, because there are three terms, this amounts to two non-redundant operators, which we decide to be the first two of the RHS of (8.2). If we now define

$$O_1^{ij} := \phi_i^2\Box\phi_j^2, \quad O_2^{ij} := \phi_i^2\phi_j\Box\phi_j, \tag{8.3}$$

then by expanding the first operator we see that

$$O_1^{ij} = 2\phi_i^2(\partial\phi_j)^2 + 2O_2^{ij}. \tag{8.4}$$

Because O_1^{ij} is made out of two non-redundant operators, it can be traded with the first non-redundant operator of (8.2), hence from now on we'll consider O_1^{ij} and O_2^{ij} the two non-redundant operators of this class of operators. With that at hand, (8.2) becomes

$$O_1^{ij} \equiv -4\phi_i\phi_j(\partial\phi_i)(\partial\phi_j), \tag{8.5}$$

and it is now apparent, because the RHS is symmetric under $i \leftrightarrow j$, that

$$O_1^{ij} \equiv O_1^{ji} \quad \Leftrightarrow \quad \phi_i^2\Box\phi_j^2 \equiv \phi_j^2\Box\phi_i^2. \tag{8.6}$$

As a final step, consider now the $i = j$ case in (8.2) and in (8.5): if we inject the result obtained from the former into the latter we obtain

$$3O_1^{ii} \equiv 4O_2^{ii} \quad \Leftrightarrow \quad 3\phi_i^2\Box\phi_i^2 \equiv 4\phi_i^3\Box\phi_i. \tag{8.7}$$

With the above last two relations we come to the conclusion that, should we be able to reduce all the operators of this class to linear combinations of O_1^{ij} and O_2^{ij}, then only $\phi_2^2\phi_1\Box\phi_1$, $\phi_1^2\phi_2\Box\phi_2$, $\phi_1^3\Box\phi_1$, $\phi_2^3\Box\phi_2$ and $\phi_1^2\Box\phi_2^2$ would be contained inside them, and therefore the latter constitute the non-redundant elements for this class of the Green's basis. This is confirmed by looking at table 8.2, where each non-total derivative operator of this class is expanded through the chain rule and, thanks to (8.2), their equivalences in terms of O_1^{ij} and O_2^{ij} operators are found. As a final remark notice that, if one sets $\phi_i = \phi_j = \phi$ in table 8.2, then one recovers back table 7.1, at it should.

Operators made of 2 fields and 4 derivatives

In order to satisfy the global symmetry $\phi_i \to -\phi_i$, operators of this class must contain either two ϕ_1 or two ϕ_2 fields. But then, by renaming $\phi_i = \phi$, for each of the two possibilities the same argument of section 7.2.2 applies, meaning that $(\Box\phi_1)^2$ and $(\Box\phi_2)^2$ are the last two elements of the Green's basis.

Obtained Green's basis

Grouping together all the operators found in the above, we arrive at the Green's basis of the $\phi^6_{1,2}$-EFT up to 6-dimensional operators in box form

$$G^{(6)}_{\phi^6_{1,2}\text{-EFT}} = \left\{(\Box\phi_1)^2, (\Box\phi_2)^2, \phi_1^3\Box\phi_1, \phi_2^3\Box\phi_2, \phi_2^2\phi_1\Box\phi_1, \phi_1^2\phi_2\Box\phi_2, \phi_1^2\Box\phi_2^2, \phi_1^6, \phi_2^6, \phi_1^2\phi_2^4, \phi_1^4\phi_2^2\right\} \tag{8.8}$$

resulting in the EFT Lagrangian

$$\begin{aligned}\mathcal{L}_{\phi^6_{1,2}\text{-EFT}} = &\sum_{i=1,2}\left[\frac{Z_{\phi_i}}{2}(\partial\phi_i)^2 - \frac{m_i^2 Z_{m_i^2}}{2}\phi_i^2\right] - \frac{C_4 Z_{C_4}}{4!}\phi_1^4 - \frac{D_4 Z_{D_4}}{4!}\phi_2^4 - \frac{K_4 Z_{K_4}}{4}\phi_1^2\phi_2^2 \\ &+ \sum_{O_j \in P^{(6)}_{\phi^6_{1,2}\text{-EFT}}} \frac{Z_{C_j} C_j}{\Lambda^2} O_j + \sum_{O_j \in \left(G^{(6)}_{\phi^6_{1,2}\text{-EFT}} \setminus P^{(6)}_{\phi^6_{1,2}\text{-EFT}}\right)} \frac{\hat{\delta}_j}{\Lambda^2} O_j.\end{aligned} \tag{8.9}$$

8.3 Obtaining the physical basis

To find the physical basis of the $\phi^6_{1,2}$-EFT, given its Green's basis, we look at the effect of a (4.11) field redefinition into the EFT Lagrangian

$$\begin{aligned}\mathcal{L}_{\phi^6_{1,2}\text{-EFT}} \to &\sum_{i=1,2}\left[\frac{Z_{\phi_i}}{2}\left(\partial\phi_i + \frac{\partial f_i}{\Lambda^2}\right)^2 - \frac{m_i^2 Z_{m_i^2}}{2}\left(\phi_i + \frac{f_i}{\Lambda^2}\right)^2\right] \\ &- \frac{C_4 Z_{C_4}}{4!}\left(\phi_1 + \frac{f_1}{\Lambda^2}\right)^4 - \frac{D_4 Z_{D_4}}{4!}\left(\phi_2 + \frac{f_2}{\Lambda^2}\right)^4 \\ &- \frac{K_4 Z_{K_4}}{4}\left(\phi_1 + \frac{f_1}{\Lambda^2}\right)^2\left(\phi_2 + \frac{f_2}{\Lambda^2}\right)^2 + (\text{OPE}) \\ &= \mathcal{L}_{\phi^6_{1,2}\text{-EFT}} + \sum_{i=1,2}\left[Z_{\phi_i}(\partial\phi_i)\frac{\partial f_i}{\Lambda^2} - m_i^2 Z_{m_i^2}\phi_i\frac{f_i}{\Lambda^2}\right] \\ &- \frac{C_4 Z_{C_4}}{3!}\phi_1^3\frac{f_1}{\Lambda^2} - \frac{D_4 Z_{D_4}}{3!}\phi_2^3\frac{f_2}{\Lambda^2} - \frac{K_4 Z_{K_4}}{2}\left(\phi_1^2\phi_2\frac{f_2}{\Lambda^2} + \phi_2^2\phi_1\frac{f_1}{\Lambda^2}\right) + (\text{OPE}),\end{aligned} \tag{8.10}$$

where again no term in the OPE get an additional field redefinition contribution. Looking at the above Lagrangian, we deduce that the only possible field redefinitions that respect both the power counting and the global symmetries are

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \Omega_{\phi^6_{1,2}\text{-EFT}} = \left\{\begin{pmatrix} \Box\phi_1 \\ \Box\phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1\phi_2^2 \\ \phi_1^2\phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1^3 \\ \phi_2^3 \end{pmatrix}\right\}, \tag{8.11}$$

therefore by (4.21) the order of the physical basis is at most five. It turns out that its order is exactly five because, as we shall show below, the $f_i = \Box\phi_i$ transformations get rid of the $(\Box\phi_i)^2$ operators, the $f_i = \phi_i\phi_j^2$ transformations get rid of the $\phi_j^2\phi_i\Box\phi_i$ operators, and the $f_i = \phi_i^3$ transformations get rid of the $\phi_i^3\Box\phi_i$ operators, for $i, j \in \{1, 2\}$ but $i \neq j$. This eventually gives us the physical basis of the $\phi^6_{1,2}$-EFT

$$P^{(6)}_{\phi^6_{1,2}\text{-EFT}} = \left\{\phi_1^2\Box\phi_2^2, \phi_1^6, \phi_2^6, \phi_1^2\phi_2^4, \phi_1^4\phi_2^2\right\}. \tag{8.12}$$

With that at hand, the physical Lagrangian is defined as

$$
\begin{aligned}
\mathcal{L}_{\phi^6_{1,2}\text{-EFT,Phys}} :=& \sum_{i=1,2}\left[\frac{Z_{\phi_i}}{2}(\partial\phi_i)^2 - \frac{m_i^2 Z_{m_i^2}}{2}\phi_i^2\right] - \frac{C_4 Z_{C_4}}{4!}\phi_1^4 - \frac{D_4 Z_{D_4}}{4!}\phi_2^4 - \frac{K_4 Z_{K_4}}{4}\phi_1^2\phi_2^2 \\
&- \frac{A_6 Z_{A_6}}{6!\Lambda^2}\phi_1^6 - \frac{B_6 Z_{B_6}}{6!\Lambda^2}\phi_2^6 - \frac{E_6 Z_{E_6}}{48\Lambda^2}\phi_1^2\phi_2^4 - \frac{F_6 Z_{F_6}}{48\Lambda^2}\phi_1^4\phi_2^2 - \frac{R_6 Z_{R_6}}{4\Lambda^2}\phi_1^2\Box\phi_2^2 \\
=& \sum_{i=1,2}\left[\frac{1}{2}(\partial\phi_{B,i})^2 - \frac{m_i^2 Z_{m_i^2}}{Z_{\phi_i}}\frac{\phi_{B,i}^2}{2}\right] - \frac{C_4 Z_{C_4}}{Z_{\phi_1}^2}\frac{\phi_{B,1}^4}{4!} - \frac{D_4 Z_{D_4}}{Z_{\phi_2}^2}\frac{\phi_{B,2}^4}{4!} \\
&- \frac{K_4 Z_{K_4}}{Z_{\phi_1} Z_{\phi_2}}\frac{\phi_{B,1}^2\phi_{B,2}^2}{4} - \frac{A_6 Z_{A_6}}{\Lambda^2 Z_{\phi_1}^3}\frac{\phi_{B,1}^6}{6!} - \frac{B_6 Z_{B_6}}{\Lambda^2 Z_{\phi_2}^3}\frac{\phi_{B,2}^6}{6!} \\
&- \frac{E_6 Z_{E_6}}{Z_{\phi_1} Z_{\phi_2}^2\Lambda^2}\frac{\phi_{B,1}^2\phi_{B,2}^4}{48} - \frac{F_6 Z_{F_6}}{Z_{\phi_1}^2 Z_{\phi_2}\Lambda^2}\frac{\phi_{B,1}^4\phi_{B,2}^2}{48} - \frac{R_6 Z_{R_6}}{\Lambda^2 Z_{\phi_1} Z_{\phi_2}}\frac{\phi_{B,1}^2\Box\phi_{B,2}^2}{4},
\end{aligned}
\tag{8.13}
$$

and as a result the EFT Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_{\phi^6_{1,2}\text{-EFT}} =& \mathcal{L}_{\phi^6_{1,2}\text{-EFT,Phys}} - \frac{\bar{\delta}_1}{2\Lambda^2}(\Box\phi_1)^2 - \frac{\bar{\delta}_2}{2\Lambda^2}(\Box\phi_2)^2 - \frac{\tilde{\delta}_1}{4!\Lambda^2}\phi_1^3\Box\phi_1 - \frac{\tilde{\delta}_2}{4!\Lambda^2}\phi_2^3\Box\phi_2 \\
&- \frac{\hat{\delta}_1}{2\Lambda^2}\phi_2^2\phi_1\Box\phi_1 - \frac{\hat{\delta}_2}{2\Lambda^2}\phi_1^2\phi_2\Box\phi_2 \\
\to& \sum_{i=1,2}\left[\frac{Z_{\phi_i}}{2}(\partial\phi_i)^2 - \frac{m_i^2 Z_{m_i^2}}{2}\phi_i^2\right] \\
&- \left(C_4 Z_{C_4} - \frac{\tilde{\delta}_1 m_1^2 Z_{m_1^2}}{\Lambda^2 Z_{\phi_1}}\right)\frac{\phi_1^4}{4!} - \left(D_4 Z_{D_4} - \frac{\tilde{\delta}_2 m_2^2 Z_{m_2^2}}{\Lambda^2 Z_{\phi_2}}\right)\frac{\phi_2^4}{4!} \\
&- \left(K_4 Z_{K_4} - \frac{2\hat{\delta}_1 m_1^2 Z_{m_1^2}}{\Lambda^2 Z_{\phi_1}} - \frac{2\hat{\delta}_2 m_2^2 Z_{m_2^2}}{\Lambda^2 Z_{\phi_2}}\right)\frac{\phi_1^2\phi_2^2}{4} \\
&- \left(\frac{A_6 Z_{A_6}}{\Lambda^2} - \frac{5\tilde{\delta}_1 C_4 Z_{C_4}}{\Lambda^2 Z_{\phi_1}}\right)\frac{\phi_1^6}{6!} - \left(\frac{B_6 Z_{B_6}}{\Lambda^2} - \frac{5\tilde{\delta}_2 D_4 Z_{D_4}}{\Lambda^2 Z_{\phi_2}}\right)\frac{\phi_2^6}{6!} \\
&- \left(\frac{E_6 Z_{E_6}}{\Lambda^2} - \frac{12\hat{\delta}_1 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_1}} - \frac{4\hat{\delta}_2 D_4 Z_{D_4}}{\Lambda^2 Z_{\phi_2}} - \frac{\tilde{\delta}_2 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_2}}\right)\frac{\phi_1^2\phi_2^4}{48} \\
&- \left(\frac{F_6 Z_{F_6}}{\Lambda^2} - \frac{12\hat{\delta}_2 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_2}} - \frac{4\hat{\delta}_1 C_4 Z_{C_4}}{\Lambda^2 Z_{\phi_1}} - \frac{\tilde{\delta}_1 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_1}}\right)\frac{\phi_1^4\phi_2^2}{48} \\
&- \left(\frac{R_6 Z_{R_6}}{\Lambda^2}\right)\frac{\phi_1^2\Box\phi_2^2}{4},
\end{aligned}
\tag{8.14}
$$

whose bare fields form takes the shape

$$
\begin{aligned}
\mathcal{L}_{\phi^6_{1,2}\text{-EFT}} \to \sum_{i=1,2}\left[\frac{1}{2}(\partial\phi_{B,i})^2 - \frac{m_i^2 Z_{m_i^2}}{Z_{\phi_i}}\frac{\phi_{B,i}^2}{2}\right] \\
- \left(\frac{C_4 Z_{C_4}}{Z_{\phi_1}^2} - \frac{\tilde{\delta}_1 m_1^2 Z_{m_1^2}}{\Lambda^2 Z_{\phi_1}^3}\right)\frac{\phi_{B,1}^4}{4!} - \left(\frac{D_4 Z_{D_4}}{Z_{\phi_2}^2} - \frac{\tilde{\delta}_2 m_2^2 Z_{m_2^2}}{\Lambda^2 Z_{\phi_2}^3}\right)\frac{\phi_{B,2}^4}{4!} \\
- \left(\frac{K_4 Z_{K_4}}{Z_{\phi_1} Z_{\phi_2}} - \frac{2\hat{\delta}_1 m_1^2 Z_{m_1^2}}{\Lambda^2 Z_{\phi_1}^2 Z_{\phi_2}} - \frac{2\hat{\delta}_2 m_2^2 Z_{m_2^2}}{\Lambda^2 Z_{\phi_1} Z_{\phi_2}^2}\right)\frac{\phi_{B,1}^2\phi_{B,2}^2}{4} \\
- \left(\frac{A_6 Z_{A_6}}{\Lambda^2 Z_{\phi_1}^3} - \frac{5\tilde{\delta}_1 C_4 Z_{C_4}}{\Lambda^2 Z_{\phi_1}^4}\right)\frac{\phi_{B,1}^6}{6!} - \left(\frac{B_6 Z_{B_6}}{Z_{\phi_2}^3\Lambda^2} - \frac{5\tilde{\delta}_2 D_4 Z_{D_4}}{\Lambda^2 Z_{\phi_2}^4}\right)\frac{\phi_{B,2}^6}{6!} \\
- \left(\frac{E_6 Z_{E_6}}{\Lambda^2 Z_{\phi_1} Z_{\phi_2}^2} - \frac{12\hat{\delta}_1 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_1}^2 Z_{\phi_2}^2} - \frac{4\hat{\delta}_2 D_4 Z_{D_4}}{\Lambda^2 Z_{\phi_1} Z_{\phi_2}^3} - \frac{\tilde{\delta}_2 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_1} Z_{\phi_2}^3}\right)\frac{\phi_{B,1}^2\phi_{B,2}^4}{48} \\
- \left(\frac{F_6 Z_{F_6}}{\Lambda^2 Z_{\phi_1}^2 Z_{\phi_2}} - \frac{12\hat{\delta}_2 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_1}^2 Z_{\phi_2}^2} - \frac{4\hat{\delta}_1 C_4 Z_{C_4}}{\Lambda^2 Z_{\phi_1}^3 Z_{\phi_2}} - \frac{\tilde{\delta}_1 K_4 Z_{K_4}}{\Lambda^2 Z_{\phi_1}^3 Z_{\phi_2}}\right)\frac{\phi_{B,1}^4\phi_{B,2}^2}{48} \\
- \left(\frac{R_6 Z_{R_6}}{\Lambda^2 Z_{\phi_1} Z_{\phi_2}}\right)\frac{\phi_{B,1}^2\Box\phi_{B,2}^2}{4}.
\end{aligned}
\tag{8.15}
$$

In the above, the arrow symbolizes the three successive field redefinitions

$$
\begin{aligned}
&\mathbf{f} = \begin{pmatrix} a_1\Box\phi_1 \\ a_2\Box\phi_2 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} b_1\phi_1\phi_2^2 \\ b_2\phi_1^2\phi_2 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} c_1\phi_1^3 \\ c_2\phi_2^3 \end{pmatrix}, \\
&\text{with} \qquad a_i = -\frac{\bar{\delta}_i}{2Z_{\phi_i}}, \qquad b_i = -\frac{\hat{\delta}_i}{2Z_{\phi_i}}, \qquad c_i = -\frac{\tilde{\delta}_i}{4!Z_{\phi_i}}.
\end{aligned}
\tag{8.16}
$$

The calculations for the first and last field redefinitions are totally analogous to those done in the previous Chapter. For the second transformation, using (8.2), we have that for $i \neq j$

$$
Z_{\phi_i}(\partial\phi_i)\frac{\partial(\phi_i\phi_j^2)}{\Lambda^2} = -\frac{\hat{\delta}_i}{2\Lambda^2}\left[(\partial\phi_i)^2\phi_j^2 + 2\phi_i\phi_j(\partial\phi_i)(\partial\phi_j)\right] \equiv \frac{\hat{\delta}_i}{2\Lambda^2}\phi_j^2\phi_i\Box\phi_i. \tag{8.17}
$$

Later in this Chapter, it will turn out that $\bar{\delta}_i = 0$, therefore the effects of the first field redefinition weren't included in (8.15). Notice that we kept the arbitrary 4! factors for $\tilde{\delta}_i$, instead of correcting them to 3!, because that way we exactly recover two decoupled ϕ^6-EFTs (7.40) if we set all the additional coupling constants K_4, E_6, F_6, R_6 and all the extra improper counterterms $\hat{\delta}_1, \hat{\delta}_2$ to zero. In this limit, D_4 becomes the C_4-like interaction among ϕ_2 fields, A_6 becomes the C_6-like interaction among ϕ_1 fields and B_6 becomes the C_6-like ϕ_2-version of it. Moreover, unlike in $P^{(6)}_{\phi^6\text{-EFT}}$, here we see that the first element of the physical basis still contains a box operator, meaning that its Feynman rule will be momentum-dependent. Although the rank-4 tensor reduction rules (5.24) and (5.25) can handle it as a consequence of (4.8) and (4.10), we would have preferred that said box operator didn't appear in the physical basis. Indeed, that would have spared us lengthy tensor reductions, accentuated by the high amount of interaction vertices in $P^{(6)}_{\phi^6_{1,2}\text{-EFT}}$. However that is simply not possible because, as we'll immediately see below, by exhaustion there doesn't exist a field redefinition to reduce it into combinations of all the other operators. To see it more explicitly, if the $\phi_i^2\Box\phi_j^2$ operators were redundant, then by (8.4) we would need to reduce $\phi_i^2(\partial\phi_j)^2$ operators.

To remove them, looking at its effect on $\mathcal{L}_{\phi^6_{1,2}\text{-EFT}}$, we would need a (4.11) transformation satisfying

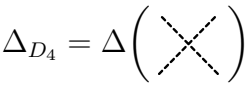
$$\begin{aligned}(\partial^\mu\phi_i)(\partial_\mu f_i) &\overset{!}{\sim} \phi_j^2(\partial^\mu\phi_i)(\partial_\mu\phi_i)\\ \Rightarrow \forall\mu\,:\, f_i &\sim \int \mathrm{d}^d x^\mu\, \phi_j^2(\partial_\mu\phi_i),\end{aligned} \tag{8.18}$$

which respects the power counting and is Lorentz invariant, but which crucially doesn't admit a closed form. The above can be explicitly evaluated only in the $\phi_i = \phi_j = \phi$ case, which corresponds to the ϕ^6-EFT, giving us the familiar $f \sim \phi^3$ transformation that was removing $3\phi^2\Box\phi^2 \equiv 4\phi^3\Box\phi$ operators (7.30).

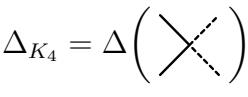
8.4 Renormalization

We now determine which 1PI n-point functions diverge. By computing the indices of divergence of all the interaction vertices present in the theory

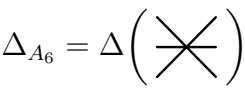
$$\Delta_{C_4} = \Delta\left(\quad\right) = -2\varepsilon = -[C_4], \tag{8.19}$$



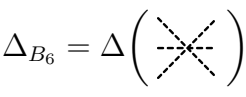
$$\Delta_{D_4} = \Delta\left(\quad\right) = -2\varepsilon = -[D_4], \tag{8.20}$$



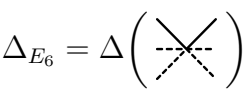
$$\Delta_{K_4} = \Delta\left(\quad\right) = -2\varepsilon = -[K_4], \tag{8.21}$$



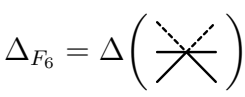
$$\Delta_{A_6} = \Delta\left(\quad\right) = 2 - 4\varepsilon = -[A_6/\Lambda^2], \tag{8.22}$$



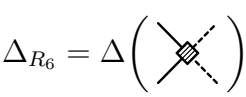
$$\Delta_{B_6} = \Delta\left(\quad\right) = 2 - 4\varepsilon = -[B_6/\Lambda^2], \tag{8.23}$$



$$\Delta_{E_6} = \Delta\left(\quad\right) = 2 - 4\varepsilon = -[E_6/\Lambda^2], \tag{8.24}$$



$$\Delta_{F_6} = \Delta\left(\quad\right) = 2 - 4\varepsilon = -[F_6/\Lambda^2], \tag{8.25}$$



$$\Delta_{R_6} = \Delta\left(\quad\right) = 2 - 2\varepsilon = -[R_6/\Lambda^2], \tag{8.26}$$

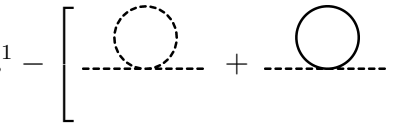
we deduce that at $d = 4$ the superficial degree of divergence of any diagram is given by

$$\begin{aligned}\Delta &= 4 - E_{\phi_1} - E_{\phi_2} + 2(N_{A_6} + N_{B_6} + N_{E_6} + N_{F_6} + N_{R_6}) \geq 0\\ &\Rightarrow (E_{\phi_1} + E_{\phi_2}) \leq 4 + 2(N_{A_6} + N_{B_6} + N_{E_6} + N_{F_6} + N_{R_6})\\ &\Rightarrow (E_{\phi_1} + E_{\phi_2}) \leq 6,\end{aligned} \tag{8.27}$$

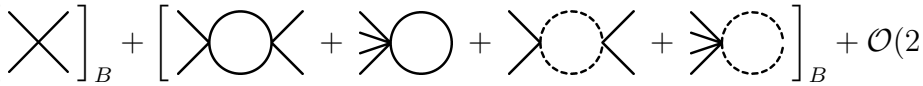
where the second implication follows from the power counting formula (4.8), that is, only one insertion of 6-dimensional operators is allowed at $\mathcal{O}(\Lambda^{-3})$. Therefore, like it was the case for the ϕ^6-EFT, we only have to renormalize the 1PI n-point functions with $n = (E_{\phi_1} + E_{\phi_2}) \in$

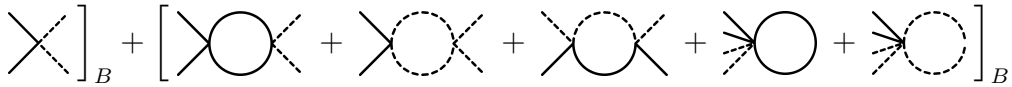
$\{2,4,6\}$. If we denote them by $\Gamma^{(n)}_{B,i_1\dots i_n} \equiv \Gamma^{(n)}_{B,\phi_{i_1}\dots\phi_{i_n}}$, then the 1PI 2-point functions are

$\Gamma^{(2)}_{B,11}(p) = ($  $)^{-1}_B -$ $+ \mathcal{O}(2\text{ loops}),$

$\Gamma^{(2)}_{B,22}(p) = ($ $)^{-1}_B -$  $+ \mathcal{O}(2\text{ loops}),$ (8.28)

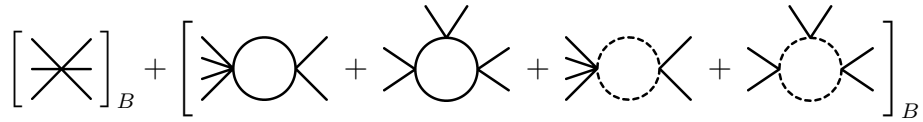
the 1PI 4-point functions are given by (sum of, respectively, 8, 6 and 8 diagrams)

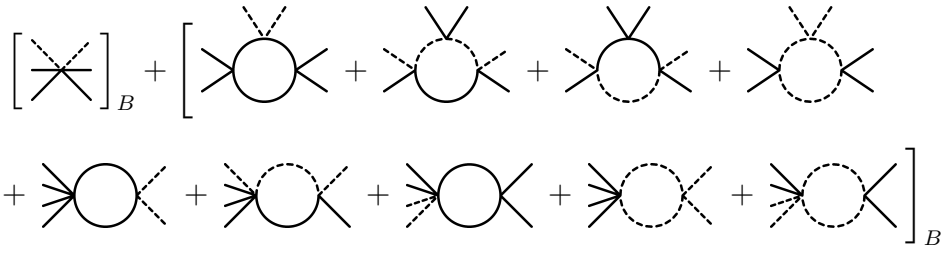
$\Gamma^{(4)}_{B,1111}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$

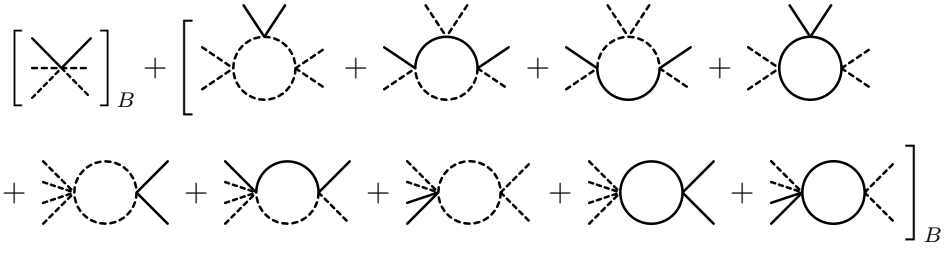
$\Gamma^{(4)}_{B,1122}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$

$\Gamma^{(4)}_{B,2222}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$ (8.29)

and the 1PI 6-point functions are given by (sum of, respectively, 60, 52, 52 and 60 diagrams)

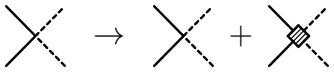
$\Gamma^{(6)}_{B,111111}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$

$\Gamma^{(6)}_{B,111122}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$

$\Gamma^{(6)}_{B,112222}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$

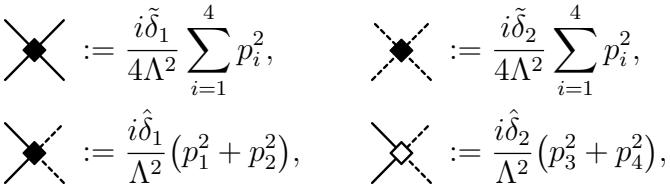
$\Gamma^{(6)}_{B,222222}(\{p_i\}) =$  $+ \mathcal{O}(2\text{ loops}),$ (8.30)

where one should substitute all the K_4-like interaction vertices with



(8.31)

and only keep the diagrams containing one insertion of the R_6 interaction vertex. Notice that, as we know from the refresher on generating functionals of Chapter 2, the ordering of the fields $i_1 \dots i_n$ in $\Gamma^{(n)}_{B,i_1 \dots i_n}$ is irrelevant. Notice also that we constrained ourselves to 1 loop diagrams: the reason for this choice is discussed further below. With all the 1PI n-point functions at hand, using the results of Chapter 6 since $m_1 = m_2 = m$, one can proceed with their renormalization. To do so, the Feynman rules of the improper counterterms are derived:



$$:= \frac{i\tilde{\delta}_1}{4\Lambda^2}\sum_{i=1}^{4} p_i^2, \qquad := \frac{i\tilde{\delta}_2}{4\Lambda^2}\sum_{i=1}^{4} p_i^2,$$

$$:= \frac{i\hat{\delta}_1}{\Lambda^2}\left(p_1^2 + p_2^2\right), \qquad := \frac{i\hat{\delta}_2}{\Lambda^2}\left(p_3^2 + p_4^2\right), \tag{8.32}$$

where, by convention, we assign to the leftmost legs the p_1 and p_2 momenta, while for the rightmost legs we assign the p_3 and p_4 momenta. We see that, for the vertices containing both fields, p_1 and p_2 are always associated to ϕ_1, while p_3 and p_4 are always associated to ϕ_2. Moreover, the Feynman rule for the R_6 momentum-dependent interaction vertex is

$$:= \frac{iR_6}{\Lambda^2}(p_3 + p_4)^2. \tag{8.33}$$

With that at hand, the computer-assisted renormalization of the $\phi^6_{1,2}$-EFT can be carried out. It turns out that, at least at 1 loop, it is $\bar{\delta}_i = \hat{\delta}_i = 0$, meaning that only $\tilde{\delta}_1, \tilde{\delta}_2$ play a role in the field redefinitions leading to the (8.15) Lagrangian. But why did we stop at 1 loop in the first place? The reason behind that decision is that, since we required the two fields ϕ_1 and ϕ_2 to have degenerate masses $m_1 = m_2 = m$ so that the algorithm of Chapter 5 could still be applied, by doing so we then actually lost the information of which free propagator, hence which field, was being assigned to loop momenta. While this degeneracy allows us to compute the Feynman diagrams in terms of the m mass, it is then no longer possible to revert to the formal m_1 and m_2 masses, and as a result it is no longer possible to inject the 1 loop contributions contained in $Z_{m_1^2}$ and $Z_{m_2^2}$ to proceed with the renormalization procedure past 1 loop order. However, the $m_1 = m_2 = m$ trick still permits renormalization at 1 loop, because at that level the tree level contributions amount to $\{Z_j\} = \{1 + \mathcal{O}(1 \text{ loop})\}$, so that no injection of $Z_{m_1^2}$ and $Z_{m_2^2}$ in the renormalized masses is effectively done. Another reason why we decided to stop at 1 loop is that, at 2 loops in $\Gamma^{(4)}_{1122,B}$, we found p/ε and p/ε^2 momentum-dependent poles with $p \in \{p_2^2, p_4^2\}$. Those terms are clearly not renormalizable, and they couldn't be reduced to a renormalizable form using the momenta-handling techniques described in Chapter 6. The issue also occurred in the 1PI 6-point functions, where $p = (p_i \cdot p_j)$ for $i, j \in \{1, 2, 3, 4, 5, 6\}$: this is a huge problem because, at $n = 6$, there simply doesn't exist a counterterm to remove these kinds of divergences. For all these reasons, we decided to limit ourselves to 1 loop order for the $\phi^6_{1,2}$-EFT, and by doing so the following renormalization constants and improper counterterms were obtained:

$$
\begin{cases}
Z_{\phi_1} = 1 + \mathcal{O}(2 \text{ loops}) \\
Z_{\phi_2} = 1 + \mathcal{O}(2 \text{ loops}) \\
Z_{m_1} = 1 + \dfrac{C_4'}{2\varepsilon Q} + \dfrac{K_4'}{2\varepsilon Q} + \mathcal{O}(2 \text{ loops}) \\
Z_{m_2} = 1 + \dfrac{D_4'}{2\varepsilon Q} + \dfrac{K_4'}{2\varepsilon Q} + \mathcal{O}(2 \text{ loops}) \\
Z_{C_4} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{A_6'}{C_4'} + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{F_6'}{C_4'} + \dfrac{3}{2\varepsilon Q}\dfrac{K_4'^2}{C_4'} + \dfrac{3}{2\varepsilon Q}C_4' + \mathcal{O}(2 \text{ loops}) \\
Z_{D_4} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{B_6'}{D_4'} + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{E_6'}{D_4'} + \dfrac{3}{2\varepsilon Q}\dfrac{K_4'^2}{D_4'} + \dfrac{3}{2\varepsilon Q}D_4' + \mathcal{O}(2 \text{ loops}) \\
Z_{K_4} = 1 + \dfrac{C_4'}{2\varepsilon Q} + \dfrac{D_4'}{2\varepsilon Q} + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{E_6'}{K_4'} + \dfrac{1}{2\varepsilon Q}\dfrac{m^2}{\Lambda^2}\dfrac{F_6'}{K_4'} + \dfrac{2}{\varepsilon Q}K_4' - \dfrac{8}{\varepsilon Q}\dfrac{m^2}{\Lambda^2}R_6' + \mathcal{O}(2 \text{ loops}) \\
Z_{A_6} = 1 + \dfrac{15}{2\varepsilon Q}\dfrac{F_6'K_4'}{A_6'} + \dfrac{15}{2\varepsilon Q}C_4' + \mathcal{O}(2 \text{ loops}) \\
Z_{B_6} = 1 + \dfrac{15}{2\varepsilon Q}\dfrac{E_6'K_4'}{B_6'} + \dfrac{15}{2\varepsilon Q}D_4' + \mathcal{O}(2 \text{ loops}) \\
Z_{E_6} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{B_6'K_4'}{E_6'} - \dfrac{24}{\varepsilon Q}\dfrac{D_4'K_4'R_6'}{E_6'} + \dfrac{C_4'}{2\varepsilon Q} + \dfrac{3}{\varepsilon Q}D_4' + \dfrac{3}{\varepsilon Q}\dfrac{F_6'K_4'}{E_6'} - \dfrac{24}{\varepsilon Q}\dfrac{K_4'^2R_6'}{E_6'} + \dfrac{8}{\varepsilon Q}K_4' \\
\qquad + \mathcal{O}(2 \text{ loops}) \\
Z_{F_6} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{A_6'K_4'}{F_6'} - \dfrac{24}{\varepsilon Q}\dfrac{C_4'K_4'R_6'}{F_6'} + \dfrac{D_4'}{2\varepsilon Q} + \dfrac{3}{\varepsilon Q}C_4' + \dfrac{3}{\varepsilon Q}\dfrac{E_6'K_4'}{F_6'} - \dfrac{24}{\varepsilon Q}\dfrac{K_4'^2R_6'}{F_6'} + \dfrac{8}{\varepsilon Q}K_4' \\
\qquad + \mathcal{O}(2 \text{ loops}) \\
Z_{R_6} = 1 + \dfrac{C_4'}{2\varepsilon Q} + \dfrac{D_4'}{2\varepsilon Q} + \dfrac{K_4'}{\varepsilon Q} + \mathcal{O}(2 \text{ loops}) \\
\tilde{\delta}_1' = \dfrac{4}{\varepsilon Q}K_4'R_6' + \mathcal{O}(2 \text{ loops}) \\
\tilde{\delta}_2' = \dfrac{4}{\varepsilon Q}K_4'R_6' + \mathcal{O}(2 \text{ loops})
\end{cases}
\tag{8.34}
$$

As a check, we can set $(K_4', E_6', F_6', R_6') \to \vec{\mathbf{0}}$ to readily recover the previous ϕ^6-EFT renormalization constants at 1 loop. We see that by doing so we get $\tilde{\delta}_i' = 0$, which is correct because the improper counterterms of the ϕ^6-EFT were obtained at 2 loops. The dimensionfullness of the masses is correctly cancelled by Λ^2 denominators, and we verify that the coupling constants combine accordingly to the generalization of table 6.1. As a final remark, we notice the coupling constants symmetry among the renormalization constants

$$
(C_4, A_6, F_6) \quad \leftrightarrow \quad (D_4, B_6, E_6), \tag{8.35}
$$

which is the joint consequence of the fact that $m_1 = m_2 = m$ followed by a $\phi_1 \leftrightarrow \phi_2$ permutation in the Lagrangian.

8.5 Renormalization group equations

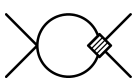
As we said in the previous section, renormalization at 1 loop is possible only because, at tree level, it is $Z_{m_1^2} = Z_{m_2^2} = 1 + \mathcal{O}(1\text{ loop})$. While this spares us from actually handling renormalized masses, a more subtle issue persists. Indeed, by letting $m_1 = m_2 = m$, it would now seem like that only one mass parameter is present in the $\phi^6_{1,2}$-EFT. However, we know that the EFT contains two distinct fields, ϕ_1 and ϕ_2, whose masses are formally also distinct, respectively m_1 and m_2. Therefore, we run once again into the issue that, given a contribution containing m, we don't know if the latter corresponds to m_1, m_2, or even a combination of the two. While this is a fundamental problem past 1 loop order, at 1 loop order it is actually possible to sneak our way out by guessing which mass m represents. Indeed, trivially 1 loop diagrams only contain one topological loop, and it can be made only in two ways: either the full loop is a single free propagator, or one half of it is ϕ_1 and the other half is ϕ_2. So, if a m-dependent divergence arises out of that loop, which we will then remove via m-dependent counterterms, we just have to figure out from which kind of loop it originated, because by transitivity the represented loop mass will also be the one landing in the counterterms. Moreover, we know that 1 loop divergences can only arise from (3.20), therefore we must only consider tadpole terms. If the loop is composed of a single ϕ_i field, then trivially $m = m_i$. Hence, during the renormalization of $\Gamma^{(4)}_{1111}$, $\Gamma^{(4)}_{2222}$ and $\Gamma^{(4)}_{1122}$, realized through, respectively, Z_{C_4}, Z_{D_4} and Z_{K_4}, the divergent diagrams constructed out of the interaction vertices of, respectively, $\{A_6, F_6\}$, $\{B_6, E_6\}$ and $\{F_6, E_6\}$, imply the following substitutions in, respectively, Z_{C_4}, Z_{D_4} and Z_{K_4}:

$$\begin{aligned} m^2 A_6 &\to m_1^2 A_6, & m^2 B_6 &\to m_2^2 B_6, & m^2 F_6 &\to m_1^2 F_6, \\ m^2 F_6 &\to m_2^2 F_6, & m^2 E_6 &\to m_1^2 E_6, & m^2 E_6 &\to m_2^2 E_6. \end{aligned} \tag{8.36}$$

If the loop is divided in half, one part being ϕ_1 and the other one being ϕ_2, as it happens in $\Gamma^{(4)}_{1122}$, then m will correspond to a combination of m_1 and m_2. To see it, first of all notice that the only way this can happen is through the R_6 interaction vertex, which is momentum dependent. Now, examine the diagram of $\Gamma^{(4)}_{1122}$ right below: if we consider the convenient case of all external momenta being null, then it is

$$\begin{aligned} &\sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\rho^2)(k^2-\sigma^2)} = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k^2-\rho^2)+(k^2-\sigma^2)+(\rho^2+\sigma^2)}{(k^2-\rho^2)(k^2-\sigma^2)} \\ &= \frac{1}{2}\left[T^D_{01} + T^D_{10} + (\rho^2+\sigma^2)T^D_{11}\right] = \frac{1}{2}\left[T^D_1 + T^D_1\Big|_{\sigma=\rho} + \frac{\sigma^2+\rho^2}{\sigma^2-\rho^2}\left(T^D_1 - T^D_1\Big|_{\sigma=\rho}\right)\right] \\ &= \frac{1}{2}\left[\frac{i\sigma^2}{\varepsilon} + \frac{i\rho^2}{\varepsilon} + \frac{\sigma^2+\rho^2}{\sigma^2-\rho^2}\left(\frac{i\sigma^2}{\varepsilon} - \frac{i\rho^2}{\varepsilon}\right) + \mathcal{O}(\varepsilon^0)\right] = \frac{2i}{\varepsilon}\frac{\sigma^2+\rho^2}{2} + \mathcal{O}(\varepsilon^0), \end{aligned} \tag{8.37}$$

where the numerator is a consequence of the R_6 Feynman rule and where in the second line we used (5.88). This has to be compared with the limiting $\rho = \sigma$ case, for which



$$\sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\sigma^2)^2} = T^D_1 + \sigma^2 T^D_2 = \frac{2i}{\varepsilon}\sigma^2 + \mathcal{O}(\varepsilon^0), \tag{8.38}$$

where the tadpole power reduction derived in (5.3.5) was applied. We see that the σ^2 mass pole in the $\rho = \sigma$ limit corresponds to $(\sigma^2+\rho^2)/2$ in the $\rho \neq \sigma$ case, therefore we conclude that we must substitute in the Z_{K_4} renormalization constant

$$m^2 R_6 \to \frac{m_1^2 + m_2^2}{2} R_6. \tag{8.39}$$

With this last substitution rule, all the previous renormalization constants are accordingly transformed, so that no more ambiguous m-dependencies are present in them:

$$\begin{cases}
Z_{\phi_1} = 1 + \mathcal{O}(2\text{ loops}) \\
Z_{\phi_2} = 1 + \mathcal{O}(2\text{ loops}) \\
Z_{m_1} = 1 + \dfrac{C'_4}{2\varepsilon Q} + \dfrac{K'_4}{2\varepsilon Q} + \mathcal{O}(2\text{ loops}) \\
Z_{m_2} = 1 + \dfrac{D'_4}{2\varepsilon Q} + \dfrac{K'_4}{2\varepsilon Q} + \mathcal{O}(2\text{ loops}) \\
Z_{C_4} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{m_1^2}{\Lambda^2}\dfrac{A'_6}{C'_4} + \dfrac{1}{2\varepsilon Q}\dfrac{m_2^2}{\Lambda^2}\dfrac{F'_6}{C'_4} + \dfrac{3}{2\varepsilon Q}\dfrac{K'^2_4}{C'_4} + \dfrac{3}{2\varepsilon Q}C'_4 + \mathcal{O}(2\text{ loops}) \\
Z_{D_4} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{m_2^2}{\Lambda^2}\dfrac{B'_6}{D'_4} + \dfrac{1}{2\varepsilon Q}\dfrac{m_1^2}{\Lambda^2}\dfrac{E'_6}{D'_4} + \dfrac{3}{2\varepsilon Q}\dfrac{K'^2_4}{D'_4} + \dfrac{3}{2\varepsilon Q}D'_4 + \mathcal{O}(2\text{ loops}) \\
Z_{K_4} = 1 + \dfrac{C'_4}{2\varepsilon Q} + \dfrac{D'_4}{2\varepsilon Q} + \dfrac{1}{2\varepsilon Q}\dfrac{m_2^2}{\Lambda^2}\dfrac{E'_6}{K'_4} + \dfrac{1}{2\varepsilon Q}\dfrac{m_1^2}{\Lambda^2}\dfrac{F'_6}{K'_4} + \dfrac{2}{\varepsilon Q}K'_4 - \dfrac{4}{\varepsilon Q}\dfrac{m_1^2+m_2^2}{\Lambda^2}R'_6 \\
\qquad + \mathcal{O}(2\text{ loops}) \\
Z_{A_6} = 1 + \dfrac{15}{2\varepsilon Q}\dfrac{F'_6K'_4}{A'_6} + \dfrac{15}{2\varepsilon Q}C'_4 + \mathcal{O}(2\text{ loops}) \\
Z_{B_6} = 1 + \dfrac{15}{2\varepsilon Q}\dfrac{E'_6K'_4}{B'_6} + \dfrac{15}{2\varepsilon Q}D'_4 + \mathcal{O}(2\text{ loops}) \\
Z_{E_6} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{B'_6K'_4}{E'_6} - \dfrac{24}{\varepsilon Q}\dfrac{D'_4K'_4R'_6}{E'_6} + \dfrac{C'_4}{2\varepsilon Q} + \dfrac{3}{\varepsilon Q}D'_4 + \dfrac{3}{\varepsilon Q}\dfrac{F'_6K'_4}{E'_6} - \dfrac{24}{\varepsilon Q}\dfrac{K'^2_4R'_6}{E'_6} + \dfrac{8}{\varepsilon Q}K'_4 \\
\qquad + \mathcal{O}(2\text{ loops}) \\
Z_{F_6} = 1 + \dfrac{1}{2\varepsilon Q}\dfrac{A'_6K'_4}{F'_6} - \dfrac{24}{\varepsilon Q}\dfrac{C'_4K'_4R'_6}{F'_6} + \dfrac{D'_4}{2\varepsilon Q} + \dfrac{3}{\varepsilon Q}C'_4 + \dfrac{3}{\varepsilon Q}\dfrac{E'_6K'_4}{F'_6} - \dfrac{24}{\varepsilon Q}\dfrac{K'^2_4R'_6}{F'_6} + \dfrac{8}{\varepsilon Q}K'_4 \\
\qquad + \mathcal{O}(2\text{ loops}) \\
Z_{R_6} = 1 + \dfrac{C'_4}{2\varepsilon Q} + \dfrac{D'_4}{2\varepsilon Q} + \dfrac{K'_4}{\varepsilon Q} + \mathcal{O}(2\text{ loops}) \\
\tilde{\delta}'_1 = \dfrac{4}{\varepsilon Q}K'_4R'_6 + \mathcal{O}(2\text{ loops}) \\
\tilde{\delta}'_2 = \dfrac{4}{\varepsilon Q}K'_4R'_6 + \mathcal{O}(2\text{ loops})
\end{cases} \tag{8.40}$$

We are now ready to inject the above renormalization constants into (8.15) and to take log-derivatives on its extended renormalization constants to generate the RGEs.

8.6 Beta functions

While our RGEs solver was (barely) able to invert the system of RGEs in a reasonable computational time, it unfortunately could never finish expanding the obtained solutions in a proper series of coupling constants, even after trying to run the former on powerful

machines. Therefore, to find the beta functions and anomalous dimensions, we used the checking equation (3.70) instead, without further verification on the obtained solutions, and eventually we got

$$
\begin{cases}
\gamma_{\phi_1} = \mathcal{O}(2 \text{ loops}) \\
\gamma_{\phi_2} = \mathcal{O}(2 \text{ loops}) \\
\beta_{m_1^2} = m_1^2\left(\dfrac{C_4'}{Q} + \dfrac{K_4'}{Q}\right) + \mathcal{O}(2 \text{ loops}) \\
\beta_{m_2^2} = m_2^2\left(\dfrac{D_4'}{Q} + \dfrac{K_4'}{Q}\right) + \mathcal{O}(2 \text{ loops}) \\
\beta_{C_4} = -2C_4'\varepsilon + \dfrac{m_1^2}{\Lambda^2}\dfrac{A_6'}{Q} + 3\dfrac{C_4'^2}{Q} + \dfrac{m_2^2}{\Lambda^2}\dfrac{F_6'}{Q} + 3\dfrac{K_4'^2}{Q} - 8\dfrac{m_1^2}{\Lambda^2}\dfrac{K_4'R_6'}{Q} + \mathcal{O}(2 \text{ loops}) \\
\beta_{D_4} = -2D_4'\varepsilon + \dfrac{m_2^2}{\Lambda^2}\dfrac{B_6'}{Q} + 3\dfrac{D_4'^2}{Q} + \dfrac{m_1^2}{\Lambda^2}\dfrac{E_6'}{Q} + 3\dfrac{K_4'^2}{Q} - 8\dfrac{m_2^2}{\Lambda^2}\dfrac{K_4'R_6'}{Q} + \mathcal{O}(2 \text{ loops}) \\
\beta_{K_4} = -2K_4'\varepsilon + \dfrac{C_4'K_4'}{Q} + \dfrac{D_4'K_4'}{Q} + \dfrac{m_1^2}{\Lambda^2}\dfrac{F_6'}{Q} + \dfrac{m_2^2}{\Lambda^2}\dfrac{E_6'}{Q} + 4\dfrac{K_4'^2}{Q} - 8\dfrac{m_1^2}{\Lambda^2}\dfrac{K_4'R_6'}{Q} - 8\dfrac{m_2^2}{\Lambda^2}\dfrac{K_4'R_6'}{Q} \\
\qquad + \mathcal{O}(2 \text{ loops}) \\
\beta_{A_6} = \dfrac{1}{\Lambda^2}\left(-4A_6'\varepsilon + 15\dfrac{A_6'C_4'}{Q} - 40\dfrac{C_4'K_4'R_6'}{Q} + 15\dfrac{F_6'K_4'}{Q}\right) + \mathcal{O}(2 \text{ loops}) \\
\beta_{B_6} = \dfrac{1}{\Lambda^2}\left(-4B_6'\varepsilon + 15\dfrac{B_6'D_4'}{Q} - 40\dfrac{D_4'K_4'R_6'}{Q} + 15\dfrac{E_6'K_4'}{Q}\right) + \mathcal{O}(2 \text{ loops}) \\
\beta_{E_6} = \dfrac{1}{\Lambda^2}\Bigg(- 4E_6'\varepsilon + \dfrac{B_6'K_4'}{Q} + \dfrac{C_4'E_6'}{Q} + 6\dfrac{D_4'E_6'}{Q} + 6\dfrac{F_6'K_4'}{Q} + 16\dfrac{E_6'K_4'}{Q} \\
\qquad\qquad - 48\dfrac{D_4'K_4'R_6'}{Q} - 56\dfrac{K_4'^2R_6'}{Q}\Bigg) + \mathcal{O}(2 \text{ loops}) \\
\beta_{F_6} = \dfrac{1}{\Lambda^2}\Bigg(- 4F_6'\varepsilon + \dfrac{A_6'K_4'}{Q} + \dfrac{D_4'F_6'}{Q} + 6\dfrac{C_4'F_6'}{Q} + 6\dfrac{E_6'K_4'}{Q} + 16\dfrac{F_6'K_4'}{Q} \\
\qquad\qquad - 48\dfrac{C_4'K_4'R_6'}{Q} - 56\dfrac{K_4'^2R_6'}{Q}\Bigg) + \mathcal{O}(2 \text{ loops}) \\
\beta_{R_6} = \dfrac{1}{\Lambda^2}\left(-2R_6'\varepsilon + \dfrac{C_4'R_6'}{Q} + \dfrac{D_4'R_6'}{Q} + 2\dfrac{K_4'R_6'}{Q}\right) + \mathcal{O}(2 \text{ loops})
\end{cases}
\tag{8.41}
$$

As a check, we can set $(K_4', E_6', F_6', R_6') \to \vec{\mathbf{0}}$ to readily recover the 1 loop beta functions and anomalous dimensions of two decoupled ϕ^6-EFTs. Moreover, we verify that the (8.35) symmetry, together with the additional $m_1 \leftrightarrow m_2$ requirement, is satisfied.

Chapter 9

Conclusions and acknowledgements

9.1 Conclusions

In this thesis, the combined efforts from analytical calculations and computer-algebra codes allowed us to semi-automatically perform the full renormalization procedure on three different field theories, namely a renormalizable theory and two EFTs, and as a consequence to quantify the running of their renormalized parameters through their obtained beta functions and anomalous dimensions.

More in detail, we first introduced known regularization and renormalization techniques to properly treat the divergences arising from the ill-defined integrals implied by Feynman diagrams. We saw that it wasn't necessary to recalculate all diagrams with the renormalized free propagator, as the more easily computable bare n-point functions could be linked to their renormalized versions. Then, the effective field theories framework was formally introduced, and a new notion of renormalizability for EFTs had to be formulated. Moreover, we saw EFTs implied the effective beta functions theorem, which simplifies enormously the determination of their beta functions because it only requires to compute the diagrams whose Feynman rules are the ones associated with the operators of their physical bases. Having developed all those theoretical tools, we then presented a symbolic evaluation algorithm of Feynman diagrams, capable to compute them in principle up to any loop order, provided one is able to completely identify all the required master integrals. In this thesis, we restrained ourselves to 1 and 2 loops diagrams, with the additional requirement of common masses among different fields. With that at hand, we ran the computer-algebra code to determine the complete list of Feynman diagrams of ϕ^4-theory and of the ϕ^6-EFT up to 2 loops. Said list was also partially verified by hand. Moreover, we introduced and proved a symbolic loop order counting technique, which turned out to be very useful to determine which contributions had to be truncated away once we started manipulating the renormalization constants. That happened as soon as we began renormalizing ϕ^4-theory and the ϕ^6-EFT, because in the latter multiple coupling constants started to appear, likewise in the $\phi^6_{1,2}$-EFT. The first two of these three theories were fully renormalized up to 2 loops, while the last one up to 1 loop. The known ϕ^4-theory was also verified in parallel by hand. For the two EFTs, we determined their Green's bases and, through field redefinitions, also their physical bases. Eventually, all the beta functions and anomalous dimensions were found, and it was observed that, since the ϕ^6-EFT is a generalization of ϕ^4-theory and the $\phi^2_{1,2}$-EFT is a generalization of the ϕ^6-EFT, their beta functions and anomalous dimensions reduced to those of their previous less general theory if one appropriately set their additional coupling constants to zero.

Further expansion of this work could be undertaken in both the difficulties encountered in this thesis and in more general considerations. Regarding the problems encountered in

this thesis, the first issue we came across was that we couldn't find the appropriate power reduction formulas to reduce the $H^D_{\nu_1,\nu_2,\nu_3}$ scalar integrals. A possible solution could be to implement IBP tables instead of trying to find ways to analytically chain the IBPs such to obtain power reduction analytical formulas. That way, the power reduction would be done by a repeated lookup of the correct descent path in said precalculated tables. The second problem we encountered was that, for the computer-algebra code, 1 loop diagrams made out of two fields with different masses had momentum-dependent divergences, even if all their momentum-dependent contributions implied by (4.10) had their superficial degree of divergence $\Delta < 0$ and these bits of integral were clearly convergent. This led us to forbid theories having different masses for different fields. A possible solution to this problem would be to abandon the symbolic evaluation algorithm altogether, to instead rely on the alternative method of equation (5.92), which doesn't even require knowledge of master integrals. Even more drastically, one could simply implement the ε-expansion of the general sunset analytical formula derived in reference [14], giving instant access to the value of tensor-reduced integrals. The third issue we encountered was that the 2 loops contributions of the 1PI n-point functions of the $\phi^2_{1,2}$-EFT contained momentum-dependent poles which had an unnormalizable form, as there doesn't exist a counterterm to cancel them. Moreover, we couldn't verify if said poles would cancel after injection of the renormalization constants containing the 1 loop renormalization counterterms because, since both ϕ_1 and ϕ_2 have the same masses $m_1 = m_2 = m$, then the information of which free propagator, hence which mass, was used in internal loops was lost, and this made it impossible to account for the renormalization of these two masses past 1 loop order. The obvious solution to this problem would be to allow $m_1 \neq m_2$, for which the previous considerations apply. Finally, the last problem we encountered was that the system of RGEs of the $\phi^6_{1,2}$-EFT was almost too big to be inverted, but definitely too big for its inversion to be expanded in a series of coupling constants. Because that forced us to employ the checking equation (3.70) to actually determine the beta functions, a new test has to be devised: a possible candidate could be the cumbersome more general checking equation we derived in Chapter 3.

Regarding more general considerations, as a first extension of this work, we could verify the effective beta functions theorem by actually computing the beta functions and the anomalous dimensions considering the diagrams obtained with the Feynman rules assigned to the operators of the Green's basis instead. This implies providing a renormalization constant to every operator of the Green's basis, as opposed to the use of improper counterterms, so that field redefinitions would not be necessary anymore. However, as we saw, the elements of the Green's bases often contained box operators, resulting in momentum-dependent Feynman rules, which would require higher-rank tensor reduction rules. As a second extension, we could have completely automated the whole renormalization procedure, without asking for human intervention at various points, such that renormalization could be carried out given any arbitrary Lagrangian, provided by the user as an input. We are confident that, with the current state of our computer-algebra codes, this task simply amounts to developing a logistical program which would extract results from one script to supply them to another one. Moreover, the determination of the Green's and physical bases could have been automated, too. As a third extension, we could have handled the EFT cutoff prescription differently. In this thesis, the (4.10) expansion was indiscriminately executed for any n-point function, however the former should have been truncated in function of the value of n. Indeed, if for example we consider 6-dimensional OPEs and we symbolically denote by p the external momentum dependence of 1PI n-point functions contributions, then it is clear that p^4-divergences could be renormalized in 1PI 2-point functions, p^2-divergences in 1PI 4-point functions and p^0-divergences in 1PI 6-point functions. This last point is important: we were

lucky that there weren't any momentum-dependent poles in the 1PI 6-point functions of the ϕ^6-EFT at 2 loops order and of the $\phi^6_{1,2}$-EFT at 1 loop order, but had we found them, then we could have simply set $p = 0$ to remove them, and that is because there's simply no operator in the Green's basis able to renormalize these kinds of divergences, namely no box operators can appear together with 6 fields in 6-dimensional OPEs. Lastly, we obviously could have considered a higher loop order, for example diagrams up to 3 loops, and higher-dimensional OPEs, for example operators up to dimension 8. Other than requiring more extenuating calculations, in principle the formalisms presented in this thesis could very well handle more general cases.

It would also be interesting to allow for fermions and massless gauge bosons, so that all the elements would be present to renormalize the SMEFT at 2 loops. We are confident that such a task it within the reach of the techniques outlined in this thesis.

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Appendix A

Semi-general sunset derivation

As in the sunset integral derivation (3.22), start with

$$\begin{aligned}
H^D_{111} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{k^2-\rho^2}\frac{1}{l^2-\sigma^2}\frac{1}{(k-l)^2-\sigma^2} \\
&= \Gamma(3)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{\left(x_{13}k^2+x_{23}l^2-\sigma^2-x_1(\rho^2-\sigma^2)-2x_3(k\cdot l)\right)^3} \\
&= \Gamma(3)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{\left(x_{13}k^2+x_{23}l^2-\tilde{\sigma}^2-2x_3(k\cdot l)\right)^3} \\
&= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\int_0^1 \mathrm{d}\mathbf{x}\,\tilde{\sigma}^{2-4\varepsilon}(ab)^{\varepsilon-2}\delta(1-x_{123}) \\
&= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\sigma^{2-4\varepsilon}\int_0^1 \mathrm{d}\mathbf{x}\,(1+sx_1)^{1-2\varepsilon}(ab)^{\varepsilon-2}\delta(1-x_{123}),
\end{aligned}$$

where in the second line we again go into Feynman representation (with unity power indices) and we substitute $x_{123}=1$ because of the delta function, in the third line we defined the new mass $\tilde{\sigma}^2=\sigma^2(1+sx_1)$ with $s=(\rho/\sigma)^2-1$, in the fourth line we adapted the result of the previous calculation (3.22) using the same change of variables and auxiliary variables a and b, and in the last line we brought back the original σ mass together with the additional s parameter. Clearly, by setting $s=0$ hence $\rho=\sigma$, at any step of this derivation we would revert to the previous derivation of the sunset integral. Now, what is left to determine is an integration over $\mathbf{x}$ that is computed as follows:

$$\begin{aligned}
&\int_0^1 \mathrm{d}\mathbf{x}\,(1+sx_1)^{1-2\varepsilon}\frac{\delta(1-x_{123})}{(ab)^{2-\varepsilon}} = \\
&= \int_0^1 \mathrm{d}\mathbf{x}\,\delta(1-x_{123})\frac{(1+sx_1)^{1-2\varepsilon}+(1+sx_2)^{1-2\varepsilon}+(1+sx_3)^{1-2\varepsilon}}{3(x_1x_2+x_2x_3+x_1x_3)^{2-\varepsilon}} \\
&= 2\int_0^1 \mathrm{d}\mathbf{x}\,\delta(1-x_{123})\theta(x_1>x_2)\theta(x_2>x_3)\frac{(1+sx_1)^{1-2\varepsilon}+(1+sx_2)^{1-2\varepsilon}+(1+sx_3)^{1-2\varepsilon}}{(x_1x_2+x_2x_3+x_1x_3)^{2-\varepsilon}} \\
&= 2\int_0^1 \mathrm{d}\mathbf{u}\left(u_1^2u_2\right)^{\varepsilon-1}\delta[1-u_1(1+u_2(1+u_3))]\cdot \\
&\quad\cdot\frac{(1+su_1)^{1-2\varepsilon}+(1+su_1u_2)^{1-2\varepsilon}+(1+su_1u_2u_3)^{1-2\varepsilon}}{(1+u_3(1+u_2))^{2-\varepsilon}} \\
&= 6\int_0^1 \mathrm{d}u_2\,\frac{1}{u_2^{1-\varepsilon}}\int_0^1 \mathrm{d}u_3\,\frac{1}{3}\frac{[1+u_2(1+u_3)]^{1-2\varepsilon}}{[1+u_3(1+u_2)]^{2-\varepsilon}}\cdot
\end{aligned}$$

$$\cdot \left[\left(1 + \frac{s}{1+u_2(1+u_3)}\right)^{1-2\varepsilon} + \left(1 + \frac{su_2}{1+u_2(1+u_3)}\right)^{1-2\varepsilon} + \left(1 + \frac{su_2u_3}{1+u_2(1+u_3)}\right)^{1-2\varepsilon} \right]$$
$$= 6 \int_0^1 \mathrm{d}u_2 \, \frac{1}{u_2^{1-\varepsilon}} \Big\{ [F(u_2) - F(0)] + F(0) \Big\},$$

where in the second line we symmetrized the integral because the denominator of the integrand is invariant under permutations of its Feynman parameters, in the third line we multiplied the whole integrand with the tautology (3.26), which produced a factor of $|S_3| = 3! = 6$ again because now the integrand is invariant under $\mathbf{x}$ variables permutations, in the fourth line the Heaviside functions are implemented as a change of variables (3.27), in the fifth line the u_1-integration is performed to remove the delta function (using its scaling property), and in the sixth line we defined

$$F(u_2) := \int_0^1 \mathrm{d}u_3 \, \frac{1}{3} \frac{[1+u_2(1+u_3)]^{1-2\varepsilon}}{[1+u_3(1+u_2)]^{2-\varepsilon}} \cdot$$
$$\cdot \left[\left(1 + \frac{s}{1+u_2(1+u_3)}\right)^{1-2\varepsilon} + \left(1 + \frac{su_2}{1+u_2(1+u_3)}\right)^{1-2\varepsilon} + \left(1 + \frac{su_2u_3}{1+u_2(1+u_3)}\right)^{1-2\varepsilon} \right].$$

Again, we see that for $\varepsilon \neq 1$ it is

$$F(0) = \int_0^1 \mathrm{d}u_3 \, \frac{(1+s)^{1-2\varepsilon} + 2}{3(1+u_3)^{2-\varepsilon}} = \frac{1-2^{\varepsilon-1}}{1-\varepsilon} \frac{(1+s)^{1-2\varepsilon}+2}{3} \xrightarrow{\varepsilon\to 0} \frac{3+s}{6},$$

hence it can be analytically computed through `Mathematica` that

$$\int_0^1 \mathrm{d}u_2 \, \frac{F(u_2) - F(0)}{u_2^{1-\varepsilon}} \bigg|_{\varepsilon=0} = \frac{3+s}{6} \ln(2) < \infty.$$

Since the above is finite for $\varepsilon = 0$, it must also be finite for any value of ε up to $\mathcal{O}(\varepsilon)$ additional terms. This means, like in the previous derivation, that the first part of the integral constitutes its finite part, while it is the second part that provides its diverging part

$$\int_0^1 \mathrm{d}\mathbf{x} \, (1+sx_1)^{1-2\varepsilon} \frac{\delta(1-x_{123})}{(ab)^{2-\varepsilon}} =$$
$$= 6 \left[\left(\frac{3+s}{6} \ln(2) + \mathcal{O}(\varepsilon) \right) + \int_0^1 \mathrm{d}u_2 \, \frac{F(0)}{u_2^{1-\varepsilon}} \right]$$
$$= (3+s)\ln(2) + \frac{6F(0)}{\varepsilon} + \mathcal{O}(\varepsilon)$$
$$= (3+s)\ln(2) + \left(\frac{3+s}{\varepsilon} + (3+s) - (3+s)\ln(2) - 2(1+s)\ln(1+s) \right) + \mathcal{O}(\varepsilon)$$
$$= \frac{3+s}{\varepsilon} + (3+s) - 2(1+s)\ln(1+s) + \mathcal{O}(\varepsilon),$$

where in the fourth line $F(0)$ was again expanded in ε thanks to (2.6) and (2.4). Indeed, the integrand of $F(u_2)$ doesn't exhibit singularities in the $[0;1]^2 \subset \mathbb{R}^2$ integration region. With that at hand, we can finally go back to the original integral and expand once more

everything in ε to finally get

$$
\begin{aligned}
H_{111}^{D} &= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\sigma^{2-4\varepsilon}\left(\frac{3+s}{\varepsilon}+(3+s)-2(1+s)\ln(1+s)+\mathcal{O}(\varepsilon)\right)\\
&= \frac{\sigma^{2-4\varepsilon}}{(4\pi)^{d}}\Gamma(2\varepsilon-1)\left(\frac{3+s}{\varepsilon}+(3+s)-2(1+s)\ln(1+s)+\mathcal{O}(\varepsilon)\right)\\
&= -\frac{1}{2}\frac{\sigma^{2}}{(4\pi)^{4}}\left(\frac{4\pi}{\sigma^{2}}\right)^{2\varepsilon}\left(\frac{3+s}{\varepsilon^{2}}+\frac{3+s-2(1+s)\ln(1+s)}{\varepsilon}-\frac{2(\gamma-1)(3+s)}{\varepsilon}+\mathcal{O}\big(\varepsilon^{0}\big)\right)\\
&= -\frac{1}{2}\frac{\sigma^{2}}{(4\pi)^{4}}\left(\frac{4\pi}{\sigma^{2}}\right)^{2\varepsilon}\left[\frac{3+s}{\varepsilon^{2}}+\frac{9+3s-2(1+s)\ln(1+s)}{\varepsilon}+\mathcal{O}\big(\varepsilon^{0}\big)\right]\Big(1-2\varepsilon\gamma+\mathcal{O}\big(\varepsilon^{2}\big)\Big)\\
&= -\frac{1}{2}\frac{\sigma^{2}}{(4\pi)^{4}}\left(\frac{4\pi e^{-\gamma}}{\sigma^{2}}\right)^{2\varepsilon}\left[\frac{3+s}{\varepsilon^{2}}+\frac{9+3s-2(1+s)\ln(1+s)}{\varepsilon}+\mathcal{O}\big(\varepsilon^{0}\big)\right].
\end{aligned}
$$

Appendix B

General sunset derivation

Its evaluation uses exactly the same machinery previously employed in Appendix A, only with more involved calculations, therefore we'll omit most of the explanations here. As before, start with

$$
\begin{aligned}
K^D_{111} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{k^2-\rho^2}\frac{1}{l^2-\sigma^2}\frac{1}{(k-l)^2-\tau^2} \\
&= \Gamma(3)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{\left(x_{13}k^2+x_{23}l^2-x_1\rho^2-x_2\sigma^2-x_3\tau^2-2x_3(k\cdot l)\right)^3} \\
&= \Gamma(3)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{\left(x_{13}k^2+x_{23}l^2-\tilde{\sigma}^2-2x_3(k\cdot l)\right)^3} \\
&= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\int_0^1 \mathrm{d}\mathbf{x}\,\tilde{\sigma}^{2-4\varepsilon}(ab)^{\varepsilon-2}\delta(1-x_{123}) \\
&= \Gamma(3)C(3)C(\varepsilon+1)(-1)^{\varepsilon+1}\sigma^{2-4\varepsilon}\int_0^1 \mathrm{d}\mathbf{x}\,(1+sx_1+tx_3)^{1-2\varepsilon}(ab)^{\varepsilon-2}\delta(1-x_{123}),
\end{aligned}
$$

where we introduced the new mass $\tilde{\sigma}^2 = x_1\rho^2 + x_2\sigma^2 + x_3\tau^2 = \sigma^2(1+x_1 s + x_3 t)$ with $s=(\rho/\sigma)^2-1$ and $t=(\tau/\sigma)^2-1$ as control variables. Indeed, at any step of this derivation, by setting $t=0$ hence $\tau=\sigma$ one reverts to the semi-general sunset and if one additionally sets $s=0$ hence $\rho=\sigma$ one lands back to the sunset integral. The $\mathbf{x}$-integration goes as

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\mathbf{x}\,(1+sx_1+tx_3)^{1-2\varepsilon}\frac{\delta(1-x_{123})}{(ab)^{2-\varepsilon}} = \\
&= \int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})}{(x_1x_2+x_2x_3+x_1x_3)^{2-\varepsilon}}\frac{1}{6}\Big[(1+sx_1+tx_2)^{1-2\varepsilon}+(1+sx_1+tx_3)^{1-2\varepsilon} \\
&\quad +(1+sx_2+tx_1)^{1-2\varepsilon}+(1+sx_2+tx_3)^{1-2\varepsilon}+(1+sx_3+tx_1)^{1-2\varepsilon}+(1+sx_3+tx_2)^{1-2\varepsilon}\Big] \\
&= \int_0^1 \mathrm{d}\mathbf{x}\,\frac{\delta(1-x_{123})\theta(x_1>x_2)\theta(x_2>x_3)}{(x_1x_2+x_2x_3+x_1x_3)^{2-\varepsilon}}\Big[(1+sx_1+tx_2)^{1-2\varepsilon}+(1+sx_1+tx_3)^{1-2\varepsilon} \\
&\quad +(1+sx_2+tx_1)^{1-2\varepsilon}+(1+sx_2+tx_3)^{1-2\varepsilon}+(1+sx_3+tx_1)^{1-2\varepsilon}+(1+sx_3+tx_2)^{1-2\varepsilon}\Big] \\
&= \int_0^1 \mathrm{d}\mathbf{u}\,\left(u_1^2u_2\right)^{\varepsilon-1}\frac{\delta[1-u_1(1+u_2(1+u_3))]}{\left(1+u_3(1+u_2)\right)^{2-\varepsilon}}\cdot \\
&\quad \cdot\Big[(1+su_1+tu_1u_2)^{1-2\varepsilon}+(1+su_1+tu_1u_2u_3)^{1-2\varepsilon}+(1+su_1u_2+tu_1)^{1-2\varepsilon} \\
&\quad +(1+su_1u_2+tu_1u_2u_3)^{1-2\varepsilon}+(1+su_1u_2u_3+tu_1)^{1-2\varepsilon}+(1+su_1u_2u_3+tu_1u_2)^{1-2\varepsilon}\Big]
\end{aligned}
$$

$$= 6 \int_0^1 \mathrm{d}u_2 \, \frac{1}{u_2^{1-\varepsilon}} \Big\{ [F(u_2) - F(0)] + F(0) \Big\},$$

where in the last line we defined (it is set $\epsilon = 1 - 2\varepsilon$ to reduce the size of the equation)

$$\begin{aligned} F(u_2) := \int_0^1 \mathrm{d}u_3 \, \frac{1}{6} \frac{[1 + u_2(1+u_3)]^{1-2\varepsilon}}{[1 + u_3(1+u_2)]^{2-\varepsilon}} \cdot \\ \cdot \Bigg[\left(1 + \frac{s}{1 + u_2(1+u_3)} + \frac{t u_2}{1 + u_2(1+u_3)} \right)^{\epsilon} + \left(1 + \frac{s}{1 + u_2(1+u_3)} + \frac{t u_2 u_3}{1 + u_2(1+u_3)} \right)^{\epsilon} \\ + \left(1 + \frac{s u_2}{1 + u_2(1+u_3)} + \frac{t}{1 + u_2(1+u_3)} \right)^{\epsilon} + \left(1 + \frac{s u_2}{1 + u_2(1+u_3)} + \frac{t u_2 u_3}{1 + u_2(1+u_3)} \right)^{\epsilon} \\ + \left(1 + \frac{s u_2 u_3}{1 + u_2(1+u_3)} + \frac{t}{1 + u_2(1+u_3)} \right)^{\epsilon} + \left(1 + \frac{s u_2 u_3}{1 + u_2(1+u_3)} + \frac{t u_2}{1 + u_2(1+u_3)} \right)^{\epsilon} \Bigg]. \end{aligned}$$

The above integral can be easily evaluated at $u_2 = 0$: for $\varepsilon \neq 1$ it is

$$\begin{aligned} F(0) &= \int_0^1 \mathrm{d}u_3 \, \frac{2(1+s)^{1-2\varepsilon} + 2(1+t)^{1-2\varepsilon} + 2}{6(1+u_3)^{2-\varepsilon}} \\ &= \frac{1 - 2^{\varepsilon-1}}{1-\varepsilon} \frac{(1+s)^{1-2\varepsilon} + (1+t)^{1-2\varepsilon} + 1}{3} \xrightarrow{\varepsilon \to 0} \frac{3+s+t}{6}, \end{aligned}$$

hence using `Mathematica` we can analytically compute

$$\int_0^1 \mathrm{d}u_2 \, \frac{F(u_2) - F(0)}{u_2^{1-\varepsilon}} \bigg|_{\varepsilon=0} = \frac{3+s+t}{6} \ln(2) < \infty,$$

which is therefore also finite for any value of ε up to $\mathcal{O}(\varepsilon)$ additional terms. This means, like in the previous two derivations, that the first part of the integral constitutes its finite part, while it is the second part that provides its diverging part

$$\begin{aligned} &\int_0^1 \mathrm{d}\mathbf{x} \, (1 + s x_1 + t x_3)^{1-2\varepsilon} \frac{\delta(1 - x_{123})}{(ab)^{2-\varepsilon}} = \\ &= 6 \left[\left(\frac{3+s+t}{6} \ln(2) + \mathcal{O}(\varepsilon) \right) + \int_0^1 \mathrm{d}u_2 \, \frac{F(0)}{u_2^{1-\varepsilon}} \right] = (3+s+t)\ln(2) + \frac{6F(0)}{\varepsilon} + \mathcal{O}(\varepsilon) \\ &= \frac{3+s+t}{\varepsilon} + (3+s+t) - 2(1+s)\ln(1+s) - 2(1+t)\ln(1+t) + \mathcal{O}(\varepsilon), \end{aligned}$$

where in the third line $F(0)$ was again expanded in ε thanks to (2.6) and (2.4). Once again, this is expected because the integrand of $F(u_2)$ doesn't exhibit singularities in the $[0;1]^2 \subset \mathbb{R}^2$ integration region. Going back to the original integral with this result and

expanding everything in ε finally gives

$$
\begin{aligned}
K_{111}^{D} &= \frac{\sigma^{2-4\varepsilon}}{(4\pi)^{d}}\Gamma(2\varepsilon-1)\cdot \\
&\quad\cdot\left(\frac{3+s+t}{\varepsilon}+(3+s+t)-2(1+s)\ln(1+s)-2(1+t)\ln(1+t)+\mathcal{O}(\varepsilon)\right) \\
&= -\frac{1}{2}\frac{\sigma^{2}}{(4\pi)^{4}}\left(\frac{4\pi}{\sigma^{2}}\right)^{2\varepsilon}\left(\frac{3+s+t}{\varepsilon^{2}}-\frac{6\gamma+2\gamma(s+t)}{\varepsilon}\right. \\
&\quad\left.+\frac{9+3s+3t-2(1+s)\ln(1+s)-2(1+t)\ln(1+t)}{\varepsilon}+\mathcal{O}\big(\varepsilon^{0}\big)\right) \\
&= -\frac{1}{2}\frac{\sigma^{2}}{(4\pi)^{4}}\left(\frac{4\pi}{\sigma^{2}}\right)^{2\varepsilon}\Big(1-2\varepsilon\gamma+\mathcal{O}\big(\varepsilon^{2}\big)\Big)\left[\frac{3+s+t}{\varepsilon^{2}}\right. \\
&\quad\left.+\frac{9+3s+3t-2(1+s)\ln(1+s)-2(1+t)\ln(1+t)}{\varepsilon}+\mathcal{O}\big(\varepsilon^{0}\big)\right] \\
&= -\frac{1}{2}\frac{\sigma^{2}}{(4\pi)^{4}}\left(\frac{4\pi e^{-\gamma}}{\sigma^{2}}\right)^{2\varepsilon}\left[\frac{3+s+t}{\varepsilon^{2}}\right. \\
&\quad\left.+\frac{9+3s+3t-2(1+s)\ln(1+s)-2(1+t)\ln(1+t)}{\varepsilon}+\mathcal{O}\big(\varepsilon^{0}\big)\right].
\end{aligned}
$$